The Effects of Networks and Informations on Asset Pricing

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Zurich, October 21, 2009 the Dean: Prof. Dr. Dr. Josef Falkinger

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To Emily. Whom I love.

Abstract

The thesis explores the effects of buyer-supplier networks on asset pricing and merger activity and lays down the theory of external auditing. The dissertation is composed of three parts.

In the first part we propose a structural model of firm dependence in a vertically connected network of firms based on cash-flows between the buyers and suppliers. We prove that in a closed network economy the set of equivalent martingale measures depends only on the topology of the network. Network market model induces contagion effects. We develop semi-analytical formulas for corporate debt, credit default swaps and collateralized debt obligations. We test the empirical validity of the model on the subcontractors' network of the SwissAir Group. The yield spread average relative prediction error is 18% comparing to the 89% error of the Merton model.

The second part of the dissertation examines firms' merger activity and its effect on stock prices based on the risk mitigation by creating an internal capital market. We propose a solution concept for coalitional games without superadditivity, which extends the Shapley value, and apply it to the merger activity of firms in a network. The possibility of a merger increases the equity value of stand-alone firms. Higher network dependence increases merger activity and merger surplus. Increased leverage ratio generates an inverted U-shaped curve of merger activity. The merger is rarely of conglomerate type affirming previous empirical evidence and occurs predominately between either buyers or suppliers depending on which party dominates the other in number.

The third part of the thesis develops a model of optimal auditing behavior when cash flows to the firm are observed imperfectly by the outside investors. An external auditor's report produces a verifiable signal and reduces the observed cash flow volatility. Using the results in information theory we develop explicit formulas for firm's share price under auditing. The shareholders and debtholders in a firm disagree about the optimal auditing effort which primarily shields debtholders. Under sufficiently favorable economic conditions we obtain that the first best audit contract is offered irrespective of the bargaining power and the number of auditors. Finally we develop the auditing profits and firm values for a multi-unit firm which lays ground for empirical testing.

Keywords: Asset pricing, credit risk, contagion, buyer-supplier networks, network topology, mergers, coalitional games without superadditivity, optimal auditing, auditor's revenues, entropy.

JEL (2007): C02, C16, C65, G12, G13, M42, C71, G34, C71, C78.

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Chapter 1

Pricing Credit Risk in Buyer-Supplier Networks

1.1 Introduction

Generating credit spreads that are close to the observed spreads has been a challenge for the theoretical credit risk literature. Eom, Helwege, and Huang (2004) investigate the performance of several structural models[1]. They find that the models of Merton, Geske and Collin-Dufresne and Goldstein underestimate corporate yield spreads, while the model of Leland and Toft (1996) overestimates them. Longstaff and Schwartz (1995) overestimate the spreads of risky bonds and underestimate spreads on safe ones. Recently, Cohen and Frazzini (2007) observed that a long/short equity strategy based on buyer-supplier economic links of dependent firms yields yearly alphas of over 18%. Additionally, it has been widely reported that historical financial distress and defaults of dependent firms are highly correlated, c.f. Keenan (2000) and Jarrow and Yu (2001).

Based on these observations, we propose a structural model of firm dependence, which explicitly accounts for economic links, i.e. firm relationships that are excessively hard to break. The theoretical setting of economic links resembles properties of economic relationship in Grossman and Hart (1986), Hart and Moore (1990) and has its origins in the network literature; see Jackson (2005). The model is able to explain the empirical facts in Cohen and Frazzini (2007) and complements the contagion literature by giving explicit economic motives for firm contagion. We define contagion as an economic setting in which securities' prices of non directly related firms depend on the characteristics of each other. Essential to our model is the fact that the payments made by a buyer to a supplier firm lower the asset value of the buyer and increases the assets of the supplier by equal amounts.

The importance of such a buyer-supplier dependence for credit risk is supported by the following examples.

- In January 2002, the retailer K-Mart filed for the Chapter 11 bankruptcy protection. At almost the same time a supplier of K-Mart, the food distributor Fleming Companies Inc. was affected by the financial distress of K-Mart. Their shares fell by more than five percent in a single day. Besides Fleming, the retailer Footstar and Martha Stewart Living Omnimedia and Scotts Co. felt the financial distress due to K-Mart's bankruptcy.

[1] The models compared in the article of Eom, Helwege, and Huang (2004) are Merton (1974), Geske (1977), Leland and Toft (1996), Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001).

- Faced with Parmalat's bankruptcy procedures, the Danish dairy-based company Arla Food Ingredients with sales of 740 million Euros declared soon after Parmalat's default that none of its contracts were watertight.

- Cohen and Frazzini (2007) case study of Coastcast and Callaway Corp. give an example in which sales losses of the buyer firm (Callaway) affected not only the price of Callaway's stocks but in addition induced losses in the supplier firm (Coastcast), thereby influencing its stock returns.

Since economic links are expensive to break, the reduction in buy orders of one firm is likely to propagate through the whole economy. This motivates us to introduce the notion of networks, i.e. of many economically linked firms.

The model is defined in continuous time. At any point in time, there is a fixed number of firms in the model, some of which are active and others of which have defaulted. The asset process of a firm in a network evolves as follows. Each active firm issues a buy order at independent random times to all its suppliers. This action induces monetary transfers between the firms. We consider monetary transfers based on Gibrat's law (Gibrat (1931))[2] in which the firm transfer to each supplier is proportional to the supplier's firm asset value[3].

In addition to the buyer-supplier network relationships, the firms also have external sources of income which are network independent. These external sources are modeled similarly as in Merton (1974). The relationship between the external income source and the network-based income is an indicator of the dependence structure of the firm - the higher the cash flow amount generated through the external sources, the less dependent the firm is.

Firm default is modeled as a hitting time of the asset process to some exogenously determined boundary. We identify the principal debt value as the default boundary. A default in the network freezes the asset process of the defaulting firm, and the network henceforth evolves with the remaining firms. A firm that has built its relationships with only one buyer is more prone to default, compared to the firm that has diversified its business activities to multiple buyers. Since every firm depends on all other firms in a network, default contagion can be global.

We recognize that buyer-supplier relationships are not the only sources of firm dependencies. Schönbucher (2000) identifies others - direct obligor connections, dependence on the same input factors (and exposed to same price shocks), or selling to the same markets. We treat the last one to some extent by including externally generated cash flows dependence.

We prove as a first result that between default times, the firm network model can be well approximated by a diffusion process in a heavy-traffic network. This is a good approximation for competitive and low margin industries. We then assume that firms' assets are tradeable[4]. Results

[2]Gibrat's law states that the expected growth rate of firms is proportional to their sizes. If the proportionality factor is g, then the expected cash transfer that the firm receives in a time period is $g - 1$. Gibrat's law was empirically confirmed in a series of studies such as Audretsch, Klomp, Santarelli, and Thurik (2002) and Santarelli (1997) for the services industry and startup Italian firms and rejected in other papers for the manufacturing sector. For an empirical survey of the validity of Gibrat's law see the paper by Audretsch, Klomp, Santarelli, and Thurik (2002) and references and discussion therein.

[3]The same assumptions for a one or two firm example are made in Hackbarth and Morellec (2005) for capital structure and in Lambrecht and Perrudin (2003) in the case of real modeling of the firm's asset size.

[4]This holds also in an environment, where there are securities, perfectly correlated to the firms' asset processes. An extension to the case when only functions of the asset processes, such as stocks or bonds, are traded makes a model more difficult, but some of the properties of the original network model as described in this article are preserved.

about market completeness are then proven. Without external income streams, markets are incomplete. The topology of the network characterizes the set of all martingale measures - the dimension of the space is increased for every buyer-supplier chain and for every connected component of the firm network. Financial markets with external cash flows are generally complete.

Additionally, our model generates default clustering. Default clustering is a phenomenon, when a default of a major buyer causes an increased probability of suppliers' default. We compute excess mean survival times of the suppliers in a variety of networks and give economic insight about the types of networks where default clustering occurs.

We apply our model to price corporate debt, credit default swaps and collateralized debt obligations. The network model implies that the firm's risk structure can be decomposed into two components and shows how to couple them. The first component is the business relationships with immediate firm buyers. The second component is the effects of other firms in the buyer-supplier chain to which the firm belongs - the model induces contagion effects. The corporate yield spreads can therefore increase either because of firm's dependence on a limited number of buyers or because the contagion effects in the network itself are large even though the firm is diversified. A recent case of the subprime mortgage crisis is an example of the latter. We show that for normal market conditions an increase of firm network dependence increases the yield spreads of corporate bonds. This finding is in line with the empirically observed mispricings attributed to buyer-supplier dependencies in Cohen and Frazzini (2007).

Similar effects are observed for credit default swap prices with a single network firm as the referenced entity and collateralized debt obligations written on a portfolio of firms in a network. The CDO tranche yields is sensitive to two effects: the firm network dependency structure and tranche seniority. In a typical setup, firm dependency increases the CDO tranche yield, but much less than tranche seniority.

We test the empirical validity of the network model on the case of credit quality evolution of the subcontractors of the SwissAir Group (SAG). SwissAir, a subsidiary of the SAG, defaulted in October 2001. We observe that the default of SwissAir was reflected in the bank's credit ratings or that the SAG subcontractors diversified their business activities already before SwissAir default. Fitting the network pricing model to the subcontractors' bond yield data, we arrive at the mean relative error of yield spread for the largest 19 subcontractors, each of which had a business volume relation to SAG in excess of 10% is 18%, compared to the 89% mean relative error of the Merton model for this specific example.

The model of network evolution most closely related to this article is Schellhorn and Cossin (2004). They also use graph representation of the network but model firm processes explicitly as queues in a stationary environment - firms reorganize at distress by external investors injecting additional cash. Our model explicitly accounts for defaults in a network. Shin (2005) and Eisenberg and Noe (2001) consider a one period equilibrium model of price formation in a network system populated by risk-neutral agents. They derive the existence and uniqueness of prices of debt securities, but the model can be implemented only algorithmically. Our model does not account for firm strategic decisions. For a buyer-supplier chain of only two firms the reader is referred to Tirole (2003, ch. 4).

This chapter complements the contagion literature and tries to bridge the gap between the structural and reduced form models of firm default. The model identifies the risk sources coming from the buy-supply orders and at the same time shares contagion properties with the reduced form models of Jarrow and Yu (2001), Collin-Duffresne, Goldstein, and Helwege (2003) and Collin-Dufresne, Goldstein, and Hugonnier (2004). These papers consider the coupling of default intensities of two dependent firms. The paper by Bielecki, Jeanblanc, and Rutkowski (2005) derives partial differential equations for dependent firms which are difficult to handle for more than two firms. Giesecke and Weber (2006) define firm dependence statistically between neighboring nodes on a multidimensional grid. Their model differs from ours in at least two aspects. First, their model is a reduced form model; no intuition on the asset formation is given. Second, firm interaction is local but can induce global contagion effects. Our model, on the contrary, can be very general and account for dependence between arbitrary firms.

This dissertation chapter is structured as follows. The network firm dependence model is given in Section 1.2. We mark the firm dependence model to the market in Section 1.3. We analyze network market properties and derive the degree of market incompleteness. The absence of arbitrage conditions is investigated and pricing measures are characterized. Corporate and portfolio-linked securities in a network model are priced in Section 1.4. In Section 1.5 we develop the econometric setup to test the model. We apply it to predict the yield spreads of SwissAir subcontractors. Section 1.6 concludes.

1.2 Firm network dependence

We consider an infinite horizon economy where uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The filtration $\mathbb{F} := \{\mathbb{F}_t\}_{t\geq 0}$ represents the arrival of information about firms' buy orders and firms' defaults. We assume that the filtration $\mathbb{F}$ satisfies the usual conditions of right continuity and completeness with respect to $\mathbb{P}$.

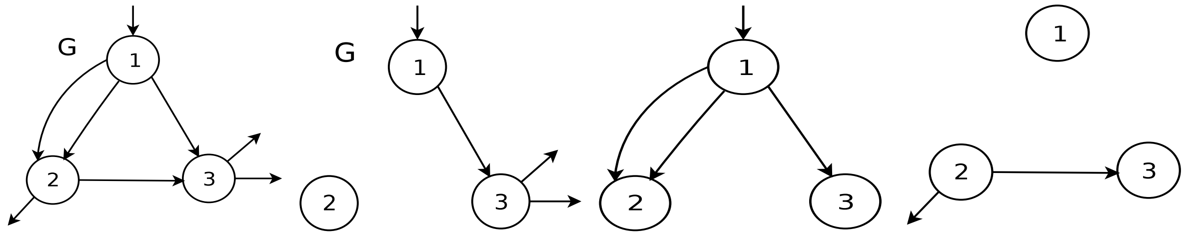
A random network is a random process $\{\mathfrak{G}\}_{t\geq 0} = \{(\mathcal{G}(t), \mathcal{E}(t))\}_{t\geq 0}$: For every $t \geq 0$, $(\mathcal{G}(t), \mathcal{E}(t))$ is a graph with nodes $\mathcal{G}(t)$, the firms, and edges $\mathcal{E}(t)$, the cash flows between the firms. The number N of firms is kept constant over time in the whole paper. The set of edges of graph $\mathfrak{G}_t$ at time t is described by an adjacency matrix $\boldsymbol{E}(t) \in \mathbb{R}^{N\times N}$, where $E_{ij}(t) \in \mathbb{F}_t$ is the number of directed connections from node i to j at time t. E_{ij} represents the extent of business relations between firms i and j. The structure of the adjacency matrix defines the topology of the network . Matrices are denoted by boldface letters and vectors are underlined in the sequel.

1.2.1 Firm asset process and adjacency matrix evolution

The firms' cash flows at time t is a vector $\underline{A}(t) \in \mathbb{R}^N$ with $A_i(t)$ the cash flow values to firm i at time t. The firms undertake the following activities: In a small time interval $\mathrm{d}t$ firm k issues a buy order to all of its suppliers with probability $\lambda_k \, \mathrm{d}t$. These actions are independent between all firms. Conditionally on a buy order, the amount $E_{kj}P_jA_j$ is paid to the supplier j where P_j is a normalization factor. This factor measures the relative strength of business dependencies across firms and simplifes the analysis of a network market by allowing for integer values E_{ij}. We set $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)'$ for the buy order intensities where P_j are proportionality factors.

Besides these network internal cash flows, exogenous cash flows are considered too, i.e. cash flows which are linked only to a single node in the network. The consideration of exogenous cash flows gives us the flexibility to model statistically cash flows to an open network and it also enables us to partition a large network into smaller ones of particular interest. External cash flows can be either positive or negative. We write B_i for the cash amount of external cash flows to firm i. As an example, consider firm 1 (node 1) which buys twice the amount of goods from firm 2 than from firm

3, see Figure 1.1(a). This is modelled by introducing twice as many connections (arrows) between firms 1 and 2 as between firms 1 and 3. Sources of the firms' external cash flows have no nodes. Firm 1 buys goods from all firms, while firm 3 is the supplier of goods to both firm 1 and 2. Following a



(a) Complete network. (b) Default of firm two. (c) Payments by firm one. (d) Payments by firm two.

Figure 1.1: A network G of three firms $(1, 2, 3)$ with external sources of cash flows (arrows not emanating from any node) is presented in 1.1(a). The network of firms after the default of firm two (Figure 1.1(b)). The cash payments by firm 1 (Figure 1.1(c)) and firm 2 (Figure 1.1(d)).

buy order, the asset values of the supplier firms change. We assume that the nominal amount of the buy order exactly offsets the good's value while the supplier earns a strictly net positive flow. There is no strategic interaction on the firms' side, nor is there an explicit input-output price modelling in the economy. We next define the dynamics for the asset/connection process $V = (\underline{A}, \boldsymbol{E})$.

Dynamics of the asset process $\underline{A}$

We assume that A_i ($i = 1, \ldots, N$) satisfies the dynamics

$$\mathrm{d}A_i = A_i \left(\sum_{j=1}^{N} E_{ij} P_i \,\mathrm{d}N_j + B_i P_i \,\mathrm{d}N_i + \eta_i \,\mathrm{d}t \right), \tag{1.1}$$

where $\underline{N} = (N_1, \ldots, N_N)'$ is a vector of independent Poisson processes with intensities $(\lambda_1, \ldots, \lambda_N)$ and η_i is the drift of A_i. The dynamics of all cash flows reads

$$d\underline{A}(t) = \boldsymbol{A}(t-) \left[\boldsymbol{P}(t-)(\boldsymbol{E}(t-) + \boldsymbol{B}(t-)) \,\mathrm{d}\underline{N}(t) + \underline{\eta} \,\mathrm{d}t \right], \qquad \underline{A}(0) = \underline{A}_0 \tag{1.2}$$

where we have denoted by $\boldsymbol{A} = \mathrm{diag}(A_1, \ldots, A_N)$, $\boldsymbol{P} = \mathrm{diag}(P_1, \ldots, P_N)$ and $\boldsymbol{B} = \mathrm{diag}(B_1, \ldots, B_N)$ the diagonal matrices with elements A_i, P_i and B_i on their diagonals respectively. The existence of a strong solution to the stochastic differential equation (1.2) follows from Theorem 1.19, Øksendal and Sulem (2005). Several assumptions are made in (1.2). First, we assume that the cash flows to the firm upon an issue of a buy order are proportional to its size. This is supported empirically in Gibrat's Law (Gibrat (1931)). Second, the exogenous cash flows $A_i P_i B_i$ to firm i arrive at the same time (and with the same intensity) as the payment streams to i-th suppliers, that is the exogenous cash flows to firm i and the payments of firm i to i-th suppliers are perfectly correlated.

Given the cash flow dynamics (1.2), the present value I_i of firm's i assets is given by

$$I_i(t) = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\infty} e^{-r(s-t)} \,\mathrm{d}A_i(s) \right] \tag{1.3}$$

for a constant discount factor r and for some exogenously defined equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$. By the Girsanov theorem for Poisson processes (see Jeanblanc, Yor, and Chesney (2005)) all equivalent martingale measures changes are given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = L(t), \tag{1.4}$$

where L is the exponential martingale defined by $\mathrm{d}L(t) = L(t-)\underline{\vartheta}(t)\,\mathrm{d}\underline{M}$ and $\vartheta_i = \frac{\gamma_i}{\lambda_i} - 1$ for some vector $\underline{\gamma}$. As stated in the Proposition 9.5.2 of Jeanblanc, Yor, and Chesney (2005) under the measure change the process $\underline{N}$ is a Poisson process with intensity $\underline{\gamma}$. It is a consequence of (1.3) that the model is arbitrage-free, i.e. the process

$$e^{-rt}I_i(t) + \int_0^t e^{-rs}\,\mathrm{d}A_i(s)$$

is a $\mathbb{Q}$-martingale. The present value I_i follows the same dynamics as A_i but with a changed initial value and Poisson process intensities. This is made rigorous in the next proposition.

Proposition 1.2.1. *Let $\underline{A}$ be given as in (1.2) and assume that*

$$\boldsymbol{P}\boldsymbol{E}\underline{\gamma} + \underline{\eta} < r \cdot \underline{1}. \tag{1.5}$$

Then the dynamics for I_i follows $\frac{\mathrm{d}I_i}{I_i} = \frac{\mathrm{d}A_i}{A_i}$ where $I_i(0) = K_iA_i(0)$ for some $K_i > 0$. The factor K_i is given in the Appendix, see equation (1.18). If (1.5) holds then the model is arbitrage free also in networks with sequential firm defaults.

Since I_i and A_i follow the same dynamics we abuse the notation and write A for the present value I with the same intensities $\underline{\lambda}$ and drift $\underline{\eta}$.

Dynamics of adjacency matrix $\boldsymbol{E}$

In general, the firms' dependence structure changes over time. We assume for simplicity throughout that the network dependency structure is constant, i.e. $\boldsymbol{E}(t) = \boldsymbol{E}(0)$ for all $t \geq 0$.

We can explicitly solve (1.2) is we perform a log-normal approximation. We assume that (i) $E_{ij}P_i > -1$ and $B_iP_i > -1$ for all i, j and that (ii) $E_{ij}P_i$ and B_iP_i are small. Then, we first define

$$\tilde{E}_{ij} = \begin{cases} \log(1 + E_{ij}P_i) & i \neq j \\ \log(1 + B_iP_i) & i = j \end{cases} \tag{1.6}$$

Under the assumption (ii) we can make the approximation

$$\tilde{E}_{ij} \approx \begin{cases} E_{ij}P_i & i \neq j \\ B_iP_i & i = j \end{cases} = \boldsymbol{P} \cdot \boldsymbol{E}. \tag{1.7}$$

It follows at once, that the solution to the asset dynamics (1.1) for firm i is given by

$$A_i(t) \quad = \quad A_i(0)\exp\left(\sum_{j=1}^{N} \tilde{E}_{ij}N_j(t) + \eta_i t\right). \tag{1.8}$$

Therefore, A_i can be expressed as a Dooleans-Dode exponential (see Revuz and Yor (1991)) of a weighted sum of independent Poisson processes. This solution show that the conditions (i) are necessary to enforce positivity of the asset process $\underline{A}$.

We further consider only networks where the buyer of a good is *not* the supplier of that same good, i.e. a buyer-supplier chain has no goods transfer cycles. The model is best suited for industries which engage in product specialization, such as transformation of raw materials, or assembly lines and can accommodate completely vertically integrated networks or networks where firms depend on multiple buyers and suppliers. We call such networks *buyer-supplier chains*.

Definition 1.2.2. *A* **buyer-supplier chain** *is a network, where there is no sequence of $F_1, F_2, \ldots, F_n$ such that $E_{F_i,F_{i+1}} \neq 0$ for all $i = 1, \ldots, n-1$ and $E_{F_n,F_1} \neq 0$.*

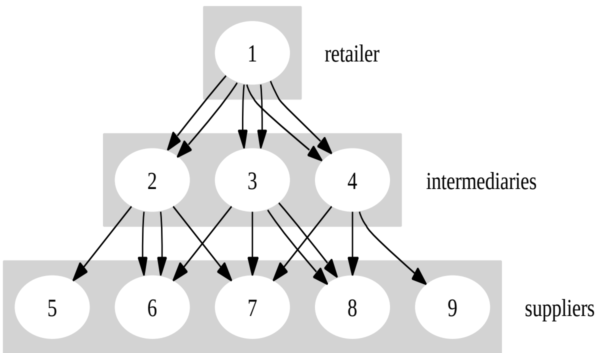


Figure 1.2: A buyer-supplier chain in tree-form. Firm 1 is the final buyer of goods and firms 5 to 9 are producers (suppliers) of initial goods. External cash flows to the firms are suppressed.

Figure 1.2 shows a buyer supplier chain with many buyers and suppliers. The adjacency matrix of this directed graph is

$$\boldsymbol{E}' = \begin{bmatrix} & 2 & 2 & 2 & & & & & \\ & & & & 1 & 2 & 1 & & \\ & & & & & 1 & 1 & 2 & \\ & & & & & & 1 & 1 & 1 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}.$$

The Proposition 1.2.3 characterizes buyer-supplier chains in terms of their adjacency matrices.

Proposition 1.2.3. *A network is a buyer-supplier chain if and only if firms can be permuted in such a way that the correspondingly permuted adjacency matrix $\boldsymbol{E}'$ is upper diagonal.*

We assume without loss of generality in the sequel that firms in the network are enumerated in the way that Proposition 1.2.3 holds.

The following proposition, which is a consequence of Theorem 1.1 in Mckay, Oggier, Royle, Sloane, Wanless, and Wilf (2004) establishes a link between the positive-definiteness of the matrix $\tilde{\boldsymbol{E}}$ and buyer-supplier chains. It states that the set of positive definite matrices $\tilde{\boldsymbol{E}}$ is in one-to one correspondence with the set of buyer-supplier chains in a network.

Proposition 1.2.4. *If $\boldsymbol{B} = \boldsymbol{0}$ then the matrix $\boldsymbol{E}$ is positive definite if and only if $\mathcal{G}$ is a buyer-supplier chain.*

Since the matrices $\tilde{\boldsymbol{E}}$ can be viewed as volatility matrices of firms' asset process values, Proposition 1.2.4 states that they can be positive definite only for networks without cycles.

The asset dynamics so far is driven by Poisson processes. But if the intensity of the network cash flows between individual firms is sufficiently large, the process A_i in (1.8) can be well approximated by a multivariate exponential Brownian motion, see Appendix 1.A.1 for precise statements. Intuitively, an asset process of a firm with many buy and supply orders occurring at a fast rate resembles a Brownian motion. This corresponds to the economy with a large number of transactions in each time interval. We refer to such economies in the sequel. The next Proposition makes the diffusion approximation to the Poisson process precise.

Proposition 1.2.5. *Let*

$$\sqrt{n}\left(\underline{\eta}+\tilde{\boldsymbol{E}}\underline{\lambda}\right)\overset{n\to\infty}{\longrightarrow}\underline{\mu}\qquad \textit{and}\qquad \tilde{\boldsymbol{E}}\boldsymbol{\lambda}^{1/2}\overset{n\to\infty}{\longrightarrow}\boldsymbol{F} \tag{1.9}$$

for some $\underline{\mu}$ and $\boldsymbol{F}$. Then A_i converges weakly to

$$A_i \overset{n\to\infty}{\longrightarrow} A_i(0)\exp\left(\sum_{j=1}^{N}F_{ij}W_j+\mu_i t\right) \tag{1.10}$$

where $(W_1,\dots,W_N)'$ is a N-dimensional Brownian motion. If furthermore $\mathcal{G}$ is a buyer-supplier chain and if for all i,k

$$\sqrt{n}E_{ij}\lambda_j \to 0 \tag{1.11}$$

then the convergence (1.10) holds also in networks with sequential firm defaults.

Conditions (1.9) are the "heavy-traffic conditions" for the network environment. They state that the cash-transfer amounts $\tilde{\boldsymbol{E}}$ have to decrease by an order of $\frac{1}{\sqrt{n}}$ to compensate for the increase in the perceived intensity of flow payments $\sqrt{n}\underline{\lambda}$. This is consistent with limit theorems for Poisson and stable processes, see e.g. Davidson (1994). The log-normal firm asset dynamics (1.8) and its limit (1.10) depend only on the dependency matrix $\boldsymbol{E}$ and not on $\boldsymbol{E}$'s higher powers.[5] Hence, in the log-normal setting only immediate dependence matters, i.e. the log-normal approximation (1.10) does not induce contagion. To induce contagion in the approximating model (1.10) we have to consider either different cash flow dynamics or dynamic firm defaults, i.e. firms default with time and only the network of remaining firms evolves further.

The last part of Proposition 1.2.5 is satisfied, if for example $\boldsymbol{E}=\boldsymbol{E}_1 f(n)$ and $\underline{\lambda}=\underline{\lambda}_1 g(n)$ for some fixed matrix $\boldsymbol{E}_1$ and $\underline{\lambda}_1$ such that $\lim_{n\to\infty} f(n)g(n)=0$. The condition is important in a network model with sequential firm defaults, see Figure 1.1(b). If a firm defaults, the network topology changes. Condition (1.11) then guarantees that the convergence formula (1.10) is valid also in a smaller network with some defaulted firms. Conditions (1.11) state that for every firm i the total amount of buy orders from every business relationship j ($E_{ji}P_i$) decreases while their intensity λ_j increases.

1.3 Financial network market model

To analyze market completeness properties of the network model in the last section, we mark the network to the market. We assume that the firms' asset process $\underline{A}$ is tradeable and introducing a riskless money market account $B(t)=B(0)e^{rt}$ with interest rate r. We refer to this construction as the *network market model*. We first specify the market prices of risk in this model.

Proposition 1.3.1. *Let the equivalent martingale measure $\mathbb{Q}\sim\mathbb{P}$ be given as in (1.4). Then the drift $\underline{\delta}$ process of $\underline{A}$ is given by*

$$\underline{\delta}=\boldsymbol{P}\boldsymbol{E}\underline{\gamma}+\underline{\eta}. \tag{1.12}$$

[5] $\boldsymbol{E}^n$ describes the dependence of a buyer on the supplier's supplier etc., i.e. there exists a network path of length n between the two depending firms.

The vector $\underline{\gamma}$ represents the market price of buy risks in a network model. The value γ_i is the elasticity of securities prices with respect to the buyers' i buy orders. The system (1.12) implies that the elasticity γ_i depends on *all* buy-supply business relationships in the network, i.e. if the firm A has a relationship with firm B then γ_A depends also on the business relationships strength E_{Bk} of firm B with other firms k. Hence, the market price of buy-risk of any one firm in a network is determined globally; that is, the model shows contagion effects[6].

Due to the change in the volatility structure of the assets after a firm default, structural models with sequential firm defaults can induce arbitrage or market incompleteness. Proposition 1.3.1 identifies conditions in which such market characteristics are not possible. The condition on B_i in Proposition 1.3.1 means that the external cash flows to every firm are non-zero (and negative) - we assume that the majority of firms' cash flows is generated by the network. This translates naturally into the condition for dynamic market completeness. A default reduces the number of firms/securities in the network and, by the correlation condition, also reduces the uncertainty associated with the buy orders. Hence, markets remain complete. We also assume that the growth rates or all firms are larger than the riskless rate, a standard assumption.

We consider the relationship between the degree of network market incompleteness, i.e. the rank of $\tilde{\boldsymbol{E}}$ and the firm network topology. We always assume in the sequel that the network market model is free of arbitrage, i.e. conditions of Proposition 1.3.1 are satisfied.

Proposition 1.3.2. *Let $\boldsymbol{B} = \mathbf{0}$ and the market be free of arbitrage, i.e. condition (1.5) holds. If the network $\mathcal{G}$ is composed of R connected components and p_i is the maximum number of disjoint paths ordered by their maximum size in component $1 \leq i \leq R$ of $\mathcal{G}$, then the dimension of the space of equivalent martingale measures to a selected $\mathbb{Q}$ of a network market model is $\sum_{i=1}^{R} p_i$. Moreover the Jordan cages of the Jordan canonical form of $\boldsymbol{E}$ corresponds to the paths in $\mathcal{G}$.*

In a network with no external cash flows, i.e. $B_i = 0$ for all firms, the degree of network market incompleteness increases with every disjoint buyer-supplier chain in the network. Therefore there exists a correspondence between the buyer-supplier chains and the degree of market incompleteness. Since every buyer-supplier network contains at least one path, the network market model without external cash flows is always incomplete. Moreover, the dimension of network market incompleteness at least equals the number of firms in the network without outgoing cash flows. The argument is as follows. Consider a firm F which is not a supplier to any other firm. Hence firm F is redundant in the network economy and adds one dimension of economic uncertainty (the buy orders) to the market, thereby increasing the dimension of market incompleteness by one. Since connected components are pairwise independent, the set of martingale measures can be defined for each component separately. This is proven in Lemma 1.A.2 of the Appendix. Hence, without loss of generality we restrict the analysis to connected networks. Finally, the network models (1.2) and (1.10), for the case $\boldsymbol{B} = \mathbf{0}$, generate the same set of equivalent martingale measures.

We claim that our model allows for default clustering, see Schönbucher (2000) for empirical evidence. To show this, we consider an example. First we use the average excess supplier survival time (MEST) after the buyer has defaulted in a single buyer - single supplier network as a measure of default clustering. Second, we set τ_1 (τ_2) for the default time of the buyer (supplier). The mean excess survival time reads $\mathbb{E}(\tau_2 - \tau_1 | \tau_2 \geq \tau_1)$. Computation of the mean excess survival time in a firm network model is similar to the pricing of path-dependent options: The default outcome of

[6]As already stated in the introduction we define contagion as an economic setting in which securities' prices of non directly related firms depend on the characteristics of each other.

any firm in a network depends on the entire asset path of all other firms. The results in Table 1.1 rely extensively on numerical computations; the theory is provided by He, Keirstead, and Rebholz (1998). The details are found in the Appendix. Table 1.1 shows the mean excess survival time of the supplier for different values of the network dependence parameter E_{12} and external cash flow levels. Columns two and three, Table 1.1, shows that in general the mean excess survival time decreases

E_{12}	MEST, $\boldsymbol{B}$ low	MEST, $\boldsymbol{B}$ high
10	0.31	0.091
15	0.29	0.087
20	0.26	0.083
25	0.23	0.077
30	0.21	0.072
35	0.19	0.067
40	0.16	0.062

Table 1.1: Mean excess survival time (MEST) for the supplier in a network of two firms for different levels of the network dependency parameter E_{12} and the level of external cash flows $\boldsymbol{B}$. All other model parameters are as in Table 1.3 reduced to the first two firms. The volatility of external cash flows in $\boldsymbol{B}^{\text{high}}$ is $B_{11}^{\text{high}} = B_{22}^{\text{high}} = -50$. Cash flow drift of both firms is 0.02. The first parameter choice (MEST, $\boldsymbol{B}$ low) corresponds to a regime of normal external cash flow volatility, approximately ranging from 5% to 20%. The second parameter choice (MEST, $\boldsymbol{B}$ high) corresponds to the highly volatile external cash flow scenario $20\% - 35\%$.

with higher buyer-supplier dependence. Low external cash flows imply higher mean excess default times while higher cash flow volatilities of external cash flows imply low excess survival time. This indicates default clustering - the higher the network dependence between the buyer and the supplier, the lower is the mean survival time of a supplier firm after the buyer firm has defaulted. Furthermore the MEST for high external cash flow dependence (column 3) are smaller than for low levels of $\boldsymbol{B}$ (column 2) indicating that large negative external cash flows reduce firm survival probabilities.

The extent of default clustering, i.e. the decrease of mean excess survival time of the supplier, is determined by a superposition of several effects. The first effect is the loss of revenues after a buyer's default. This reduces the drift of the firm's asset process which in turn lowers the mean excess survival time. Second, a buyer-supplier relationship virtually increases the drift of the supplier and has in general a positive effect on the supplier's asset distribution at the time of buyer's default. Third, there are asset volatility effects. While a highly dependent network can have a positive effect on the supplier's drift, it can have a negative effect in terms of increased asset volatility.

1.4 Pricing of securities in a network model

We apply the theory to price derivative securities in network markets. The prices depend on the network topology, i.e. the dependence which is induced by the buy and supply orders. This dependence is an additional source of risk in derivatives pricing. This source enters the pricing of all securities which we consider: zero coupon corporate debt, credit default swaps (CDS) and collateralized debt obligations (CDO) written on a portfolio of firms in a network. Without loss of generality, we consider connected networks only, see Proposition 1.3.2 and Appendix 1.A.3. To highlight the impact of the network dependence strength on prices, we use the Frobenious norm[7] of the adjacency matrix $\boldsymbol{E}$ called the *dependence value*. Intuitively, the stronger the network dependence, the larger the components E_{ij} and the higher are the Frobenious norm values. We always assume that firms can default only at maturity of their contracts. We first prove two general results which are independent of the specific securities under consideration. The first one shows that firm dependence has a global effect on securities' prices.

Proposition 1.4.1. *For every firm a with a direct path to a firm b in a network $\mathcal{G}$, the prices of firm b's securities (such as debt values, credit default swap ratios) depend on the characteristics of firm a. More precisely*

$$V_b = V_b(\eta_b, B_b, \{B_a, \eta_a, E_{ak}\}_a)$$

where V_b is the value of any of firm's b securities.

Hence the value of firm's securities depend on *all* firms to which there exists a buy-supply path and not only to the immediate buyer firms. This fact is strong evidence for firm contagion which was modelled and documented in Shin (2005). The second result states that the lower suppliers are in the buyer-supplier chain, the higher is the price of buy order risk. This is the amplification result.

Proposition 1.4.2. *If $\frac{\delta_i - \eta_i}{P_i B_i} \geq \frac{\delta_{i+1} - \eta_{i+1}}{P_{i+1} B_{i+1}} > 0$ for all $i = 1, \ldots, N-1$, $B_{i+1} \leq B_i < 0$ and $E_{i+1,j} \geq E_{ij} \geq 0$ for all $i, j = 1, \ldots, N$, then the market prices of buy risks (equation (1.12) in Proposition 1.3.1) satisfy: $\gamma_i \geq \gamma_{i+1}$.*

The assumption that $B_{i+1} \leq B_i < 0$ is justified if we assume that firms higher in the buy-supply chain have smaller external cash flows ($B_{i+1} \leq B_i$) and rely more on network generated ones. The restriction $E_{ij} \leq E_{i+1,j}$ ($j > i+1$) reflects the essential fact of the buy-supply chain that more products are sold between close successors ($i+1$ and j) in the chain than between distant ones (i and j). Under these conditions the market prices of risk of firm cash-flows are larger for firms higher in the chain than for those lower, which is another evidence for contagion in the network model - a shock to a high level supplier has a larger effect on prices for all securities connected with that supplier. Proposition 1.4.2 also identifies firms which pose a high systemic risk for the economy. It is an easy consequence of the proof of Proposition 1.4.2 that the differences $\gamma_i - \gamma_{i+1}$ are more pronounced the higher the network dependence $E_{i,i+1}$ is between them.

The network model proposed in this chapter fits in a broad class of reduced form models. If the asset process A_i is given as in (1.1) and the default event be $\tau_i = \inf\{t; A_i(t) \leq \gamma_i\}$ then the stopping time τ_i is totally inaccessible and the default time of firm i has the time-varying intensity

$$\begin{aligned}\mu_i(t) \;=\; & \lambda_i \cdot \mathbb{1}\left(\gamma_i \leq A_i(t) < \frac{\gamma_i}{1+B_i}\right) \cdot \mathbb{1}(B_i < 0) \\ & + \sum_{j \neq i}^{N} \lambda_j \cdot \mathbb{1}\left(\gamma_i \leq A_i(t) < \frac{\gamma_i}{1+E_{ij}}\right) \cdot \mathbb{1}(E_{ij} < 0) \cdot \mathbb{1}(\tau_j > t) \qquad (1.13)\end{aligned}$$

producing a recursive structure of defaults. Using the results in Collin-Dufresne, Goldstein, and Hugonnier (2004) we can reduce the pricing in the network model with intermediate defaults to the

[7] Frobenious norm $\|\boldsymbol{E}\|_F$ of matrix $\boldsymbol{E}$ is the square root of the sum of squared elements of $\boldsymbol{E}$, i.e. $\|\boldsymbol{E}\|_F = \sqrt{\sum_{i,j} E_{ij}^2}$.

computation of occupancy times of a sum of Poisson processes. We illustrate this on an example of two firm network whose default intensities then read

$$\begin{aligned}\mu_1(t) &= \lambda_1 \cdot \mathbb{1}(\gamma_1 \le A_1 < \delta_{11}) + \lambda_2 \cdot \mathbb{1}(\gamma_1 \le A_1 < \delta_{12}) \cdot \mathbb{1}(\tau_2 > t)\\ \mu_2(t) &= \lambda_1 \cdot \mathbb{1}(\gamma_2 \le A_2 < \delta_{21}) \cdot \mathbb{1}(\tau_1 > t) + \lambda_2 \cdot \mathbb{1}(\gamma_2 \le A_2 < \delta_{22})\end{aligned}$$

where we have denoted by

$$\delta_{ij} = \begin{cases} \frac{\gamma_j}{1+E_{ij}} & i \neq j \\ \frac{\gamma_i}{1+B_i} & i = j \end{cases} .$$

Using the results in Collin-Dufresne, Goldstein, and Hugonnier (2004) we can write the default probability of firm 1 by using the change of measure given in equation (4.9) in Collin-Dufresne, Goldstein, and Hugonnier (2004)

$$\begin{aligned}\mathbb{P}(\tau_1 > T|\mathbb{F}_t) &= \mathbb{1}(\tau_1 > t) \cdot \mathbb{E}^1\left[\exp\left(-\int_t^T \mu_1(s)\,\mathrm{d}s\right)|\mathbb{F}_t^1\right]\\ &= \mathbb{1}(\tau_1 > t) \cdot \Big\{\mathbb{1}(\tau_2 < t) \cdot \mathbb{E}\left[\exp\left(-\lambda_1 \int_t^T \mathbb{1}_{(-\gamma_1,\delta_{11}]}(A_1(s))\,\mathrm{d}s\right)|\mathbb{F}_t\right]\\ &\quad +\mathbb{1}(\tau_2 > t) \cdot \mathbb{E}^1\Big[\exp\Big(-\int_t^T (\lambda_1 \mathbb{1}_{(-\gamma_1,\delta_{11}]}(A_1(s))\\ &\quad +\lambda_2 \mathbb{1}_{(-\gamma_1,\delta_{12}]}(A_1(s)) \cdot \mathbb{1}(\tau_2 > s))\,\mathrm{d}s\Big)|\mathbb{F}_t^1\Big]\Big\}\end{aligned}$$

where under $\mathbb{E}^1$ the intensity of τ_2 equals

$$\lambda_1 \cdot \mathbb{1}(\gamma_2 \le A_2 < \delta_{21}) + \lambda_2 \cdot \mathbb{1}(\gamma_2 \le A_2 < \delta_{22}).$$

The network model therefore requires the computation of the Laplace transform of the occupation time of Poisson processes.

1.4.1 Pricing of corporate debt

The next result is the analogue of the classical result of Merton (1974) for corporate debt pricing in a complete network market model.

Proposition 1.4.3. *Let $N^{\lambda_1,\ldots,\lambda_N}_{a_1,\ldots a_N,T}$ be the distribution of a random variable $a_1N_1 + \ldots + a_NN_N$ where N_i are independent Poisson random variables with parameters $\lambda_1T, \ldots, \lambda_NT$ respectively. The price of a zero-coupon corporate bond of firm i in a network, with principal value D_i and maturity T_i, where all other firms are financed with securities with maturities longer than T_i is given by*

$$\begin{aligned}DV_i &= D_i e^{-rT_i}\left(1 - N^{\lambda_1,\lambda_2,\ldots,\lambda_N}_{\tilde{E}_{i1},\tilde{E}_{i2},\ldots,\tilde{E}_{iN}}(d)\right) +\\ &\quad A_i(0)e^{\left(\sum_{j=1}^N E_{ij}P_i\lambda_j - r\right)T_i} N^{(1+E_{i1}P_i)\lambda_1,(1+E_{i2}P_i)\lambda_2,\ldots,(1+E_{iN}P_i)\lambda_N}_{\tilde{E}_{i1},\tilde{E}_{i2},\ldots,\tilde{E}_{i,N}}(d) \qquad (1.14)\end{aligned}$$

where

$$d = \log\frac{D_i}{A_i(0)} - \eta_i T_i \qquad (1.15)$$

The firm i bond's hedging portfolio consists of holding

$$\underline{U}'(\boldsymbol{APE})^{-1} \tag{1.16}$$

of assets of every firm in the network where $\underline{U} = (U_1, \ldots, U_N)'$ *and*

$$U_k = DV_i(t, A_i(1 + E_{ik}P_i)) - DV_i(t, A_i).$$

To the best of authors' knowledge there does not exist an explicit expression for the distribution $N^{\lambda_1,\ldots,\lambda_N}_{a_1,\ldots,a_N,T}$. Instead we rely on the numerical inversion of the characteristic function of a weighted sum of independent Poisson distributed random variables, i.e. the probabilities $N^{\lambda_1,\ldots,\lambda_N}_{E_{i1},\ldots,E_{iN},T}$ are computed numerically by Fourier inversion. Equation (1.15) show explicitly the different sources of risk. First, there is risk associated with the exposure to every direct relationship E_{ij} weighted by the extent of relationship P_i. Second, there is a risk from the external cash flows B_i. Finally, there is risk associated to the global network dependencies λ_j which measure the exposure of the firm j to the global network economy. If the λ_i are small the risk to the firm's bond arises primarily from direct buyer-supplier relationships E_{ij}, B_i. But if the λ_i are large, this induces large effects on debt prices *even* if the immediate business relationships E_{ij} and B_i are all small.

If we do not account for a network firm dependence structure, i.e. all $E_{ij} = 0$ in (1.14) and also if $B_i = 0$, we get the result that debt is riskless: The firms have perfectly diversified their business activities.

An interesting property of the pricing model is that the drift $\underline{\eta}$ of the asset process $\underline{A}$ matters in the determination of debt prices. This observation is missing in the standard Brownian markets where the drift of the security affects the market price of risks but not the securities' prices. The reason for this difference is due to the Poisson-jumps. Since high growth firms, i.e. firms with high $\underline{\eta}$ in equation (1.2), undergo jumps in their values which are also large, their value process becomes more volatile. This then leads to higher risk premia for the corresponding buy orders.

Furthermore, as already noted in Propositions 1.4.1 and 1.4.2, the model generates contagion in the following sense. Since the market prices of buy-risks γ_i are determined endogenously by equation (1.12), the price of debt of firm i depends *not* only on its immediate buyers but also on the buyers' buyers, etc., i.e. the whole supply chain matters. More evidence on contagion is given by the hedging results (1.16). The hedging of a bond on any firm in the network, viewed as a derivative on that firm's value, is constructed by taking positions in all other firms' values in the original firm's buy-supply chain.

Theses results differ from those of related models in the contagion literature. Our model explicitly identifies the sources of risk from the network dependence contrary to the reduced form models, such as Jarrow and Yu (2001) and Collin-Dufresne, Goldstein, and Hugonnier (2004). In the latter models, authors impose the dependence of default intensities between the firms, but do not identify the risk sources.

We illustrate the relationship between overall network dependence and corporate debt prices in Figure 1.3 for the case of three firms. The model parameters are given in Table 1.3, Appendix 1.B. Under normal market conditions the network dependence increases corporate yield spreads of all three firms in the economy, see Figure 1.3. This confirms the empirical observation that Merton's model generally underestimates corporate yield spreads. It also shows that the yields of low level supplier's (firm 1) securities exhibit a much greater network dependence than the intermediary's

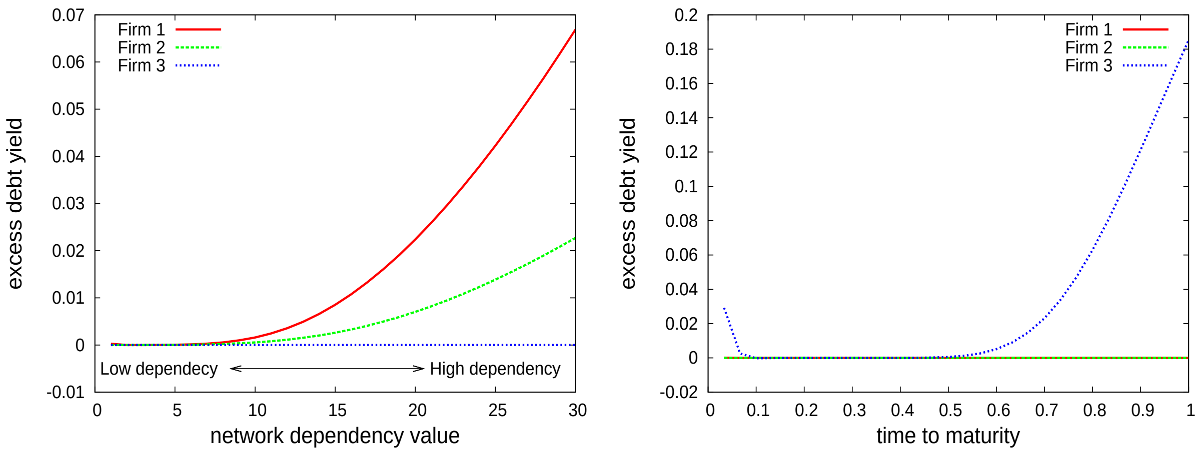


Figure 1.3: Zero-coupon debt yields in two distinct three-firm networks. The parameters are given in Table 1.3. On the horizontal axis is the network dependency value (Frobenius norm, see the beginning of Section 1.4). The volatility ranges between 0.1 and 0.2 for all firms in the network. Since $\boldsymbol{B}$ is diagonal, the only dependence between the firms arises from the network.

(firm 2) or retailer's (firm 3) securities. This effect has several sources. First, firm 1 has a stronger network dependency than the other firms. Second, there is the contagion effect which manifests itself in a higher yield slope form for firm 1 than for all other firms. We further observe that variations of the exogenous cash flow matrix $\boldsymbol{B}$ scales the excess debt yield curves of all three firms in either direction but the shape of the curves remains the same. Finally, the term structure of yield spreads in Figure 1.3 is increasing for all three firms. The highest yields arise for the firm 1 which is most exposed to the network firm and firm 3 faces the lowest yields.

Our model helps also to explain empirical facts observed in Cohen and Frazzini (2007) that there exists a positive abnormal return on stocks of mutually dependent firms. If we apply our approach to stocks of firms in the network using the same parameters as in Table 1.3, between 2% to 16% of stock return is attributed to firm network dependencies. Our model therefore offers guidance on the "pricing of economic links" between firms.

We conclude this section by proving that our model generalizes Merton's result to Poisson-type networks: As the buy-supply orders become more and more frequent (in the sense of convergence theorem in Section 1.2) we obtain the Merton's formula.

Proposition 1.4.4. *Under the conditions (1.9), the debt pricing formula (1.14)-(1.15) of Proposition 1.4.3 converges to the debt pricing formula of the Merton (1974) model with the following volatility*

$$\sigma_i^2 = B_i\sqrt{\lambda_i} + \sum_{j=i+1}^{N} E_{ij}\sqrt{\lambda_j}. \tag{1.17}$$

The result follows directly from the definition of weak convergence property in Proposition 1.2.5, equation (1.9). The network model directly relates the volatility of the firms' assets to firms' buy-supply dependence. Variance σ_i^2 exhibits a "square-root rule", i.e. the variance depends on the square root of the order transaction velocities λ_i.

1.4.2 Pricing of credit default swaps (CDS) and collateralized debt obligations (CDO)

Credit default swaps written on a reference identity in a given network are contracts between a protection buyer and a protection seller, where the protection buyer obliges himself to pay a constant amount D at times $\{t_i\}_{i=1,\dots,T}$ to the protection seller until the default of the referenced entity. The protection seller obliges himself to pay an amount U in the event of default. Without loss of generality we assume that the payments U and D are constant. The dynamics of the network firm is given in Proposition 1.3.1. We also assume that the network satisfies the conditions of Propositions 1.3.1 and 1.2.5, which insures that the network market model with sequential firm defaults is complete and free of arbitrage. The pricing formulas for a CDS are given next.

Proposition 1.4.5. *Let σ_i be defined in (1.17). We define $\alpha_i = r - \frac{1}{2}\sigma_i^2$, $\vartheta_i = \sqrt{\alpha_i^2 + 2\sigma_i^2 r}$ and $x_i = \log(A_i(0)/D_i)$. Then the ratio U/D is given by*

$$\frac{U}{D} = \frac{\overbrace{\exp\left(-\frac{x_i(\alpha_i+\vartheta_i)}{\sigma_i^2}\right) N\left(\frac{-x_i+\vartheta_i T}{\sigma_i\sqrt{T}}\right) + \exp\left(-\frac{x_i(\alpha_i-\vartheta_i)}{\sigma_i^2}\right) N\left(\frac{-x_i-\vartheta_i T}{\sigma_i\sqrt{T}}\right)}^{\text{repayment at default given default}}}{\underbrace{\sum_{j=1}^{T} e^{-rt_j}\left[N\left(\frac{x_i+\alpha_i t_j}{\sigma_i\sqrt{t_j}}\right) - \exp\left(-\frac{2\alpha_i x_i}{\sigma_i^2}\right) N\left(\frac{-x_i+\alpha_i t_j}{\sigma_i\sqrt{t_j}}\right)\right]}_{\text{discounted probability of default}}}.$$

To illustrate the result we consider the same three-firm example of the last section. The dependence of the protection seller/buyer ratio U/D on the network dependency level and CDS duration is shown in Figure 1.4. We observe as a function of the network dependence the following behavior

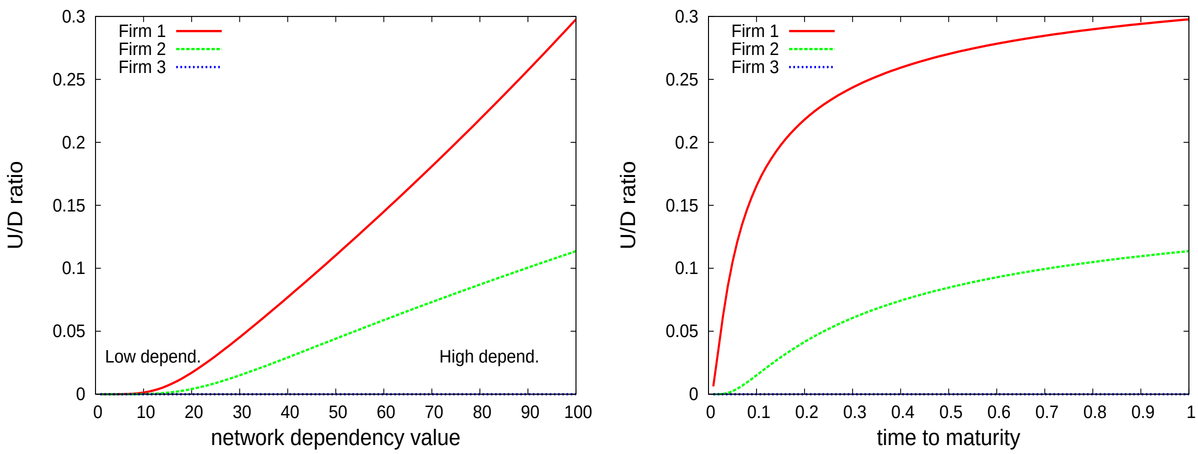


Figure 1.4: The relationship between a CDS U/D ratio and the network dependency value (defined in the introduction to Section 1.4, left figure) and the maturity of a CDS (right). The network parameters of the CDS are given in Table 1.3. For the chosen parameter values, the volatility ranges between 0.1 and 0.2 for all firms in the network. Since $\boldsymbol{B}$ is diagonal, the only dependence between the firms arises from the network.

of the U/D ratio: (a) The U/D ratio for the low-level supplier firm 1 is much higher than for the intermediary firm 2 or retailer firm 3. (b) The U/D ratio increases faster for firm 1 than firm 2. (c)

U/D ratio for firm 3 is almost zero. The first fact follows from the highest network dependence of firm 1 compared to the firm 2 or 3, i.e. the intermediary and the retailer firm are more diversified in their business activities. The second fact is a consequence of the contagion effect. Firm 1's risk is amplified by firm 2, as discussed in Proposition 1.4.2. The third fact is a consequence of the high diversification of firm 3. The term structure of CDS payments exhibits much the same effects as the network dependency structure with the absence of the characteristics of stylized fact (b) above - the network dependency structure of the network is kept constant.

CDOs are securities, where the payment to individual tranches is influenced by a network of firms. Hence, network dependence seems naturally suitable for the pricing of this kind of securities[8]. The CDOs under consideration are composed of K tranches, each tranche with principal value F_k, $k = 1, \dots, K$. Let $(D_i)_{i=1}^N$ be the individual bonds' principal payments of equal maturity T and $D = \sum_{i=1}^N D_i = \sum_{k=1}^K F_k$ the total principal payment to the bondholders. Let $C_k = \sum_{j=1}^{k-1} F_j$. The K tranches of the CDO are structured as follows. The first tranche absorbs all credit loses until the value of the remaining principal payments reaches C_{K-1}. Similarly, the i-th tranche ($i = 1, \dots, K-2$) absorbs all credit losses until the total of C_{i-1}. The K-th tranche recovers what is left. We refer to Hull and White (2005) for typical values of K and $(F_i)_{i=1}^K$.

Proposition 1.4.6. *Given the CDO structure above, the value V_k of the k-th tranche is*

$$\begin{aligned} V_k &\approx e^{-rT} F_k N(d^1_{k+1}) - C_k e^{-rT}(N(d^1_k) - N(d^1_{k+1})) + e^{-rT+m_{S_N}+\frac{1}{2}\sigma^2_{S_N}} \left(N(d^2_k) - N(d^2_{k+1})\right) \\ &= \underbrace{e^{-rT}(C_{k+1}N(d^1_{k+1}) - C_k N(d^1_k))}_{\text{tranche k repayment value}} + \underbrace{e^{-rT+m_{S_N}+\frac{1}{2}\sigma^2_{S_N}} \left(N(d^2_k) - N(d^2_{k+1})\right)}_{\text{correction term}} \end{aligned}$$

d^k_1 and d^k_2 are defined as

$$d^k_1 = \frac{m_{S_N} - \log C_k}{\sigma_{S_N}}, \qquad d^k_2 = \frac{m_{S_N} + \sigma^2_{S_N} - \log C_k}{\sigma_{S_N}}$$

with σ_i defined as in (1.17)

$$\begin{aligned} m_{S_N} &= -\frac{1}{2}\sigma^2_{S_N} + \log\left(\sum_{i=1}^N \exp\left(m_i + \frac{1}{2}\sigma_i^2\right)\right), \\ \sigma^2_{S_N} &= \log\left(1 + \frac{\sum_{i=1}^N (e^{\sigma_i^2} - 1)e^{2m_i+\sigma_i^2} + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^N e^{m_i+\frac{1}{2}\sigma_i^2} e^{m_j+\frac{1}{2}\sigma_j^2}(e^{\rho_{ij}\sigma_i\sigma_j} - 1)}{\left(\sum_{i=1}^N e^{m_i+\frac{1}{2}\sigma_i^2}\right)^2}\right), \\ m_i &= \log(A_i(0)) + \left(r - \frac{1}{2}\sum_{j=1}^N F_{ij}\right)T, \qquad \rho_{ij} = \frac{\sum_{k=1}^N F_{ik}F_{jk}}{\sigma_i\sigma_j} \end{aligned}$$

To justify the approximation results made in Proposition 1.4.6 we simulated one hundred thousand samples of sums of log-normal random variables for realistic network parameter values. The average p-value of the Kolmogorov-Smirnov statistic is around 0.4. This is a relatively high number for distributions that are not the same, but whose distribution functions are close.

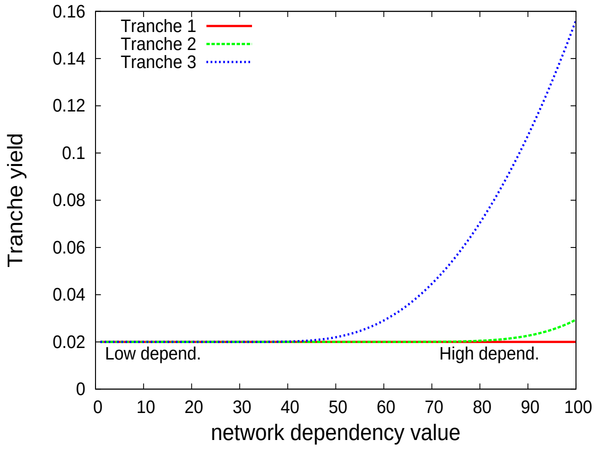


Figure 1.5: The relationship between the CDO tranche yield, the network dependency value (defined in the introduction to Section 1.4, figure left) and term structure of the CDO (right figure). The firm network parameters are given in Table 1.3. All three tranches have the same principal. For the chosen parameter values, the volatility ranges between 0.1 and 0.2 for all firms in the network. Since $\boldsymbol{B}$ is diagonal, the only dependence between the firms arises from the network.

Figure 1.5 shows the relationship between the CDO tranche yields and the network dependency value. Tranche prices in Figure 1.5 exhibit similar behavior as the corresponding corporate debt prices and CDS U/D ratios in Figures 1.3 and 1.4. Tranche yields increase with increased degree of firm network dependencies. Tranches' risk sharing effect decreases the value of the junior tranche 3 much more (and with greater intensity) than the senior tranches 1 and 2.

1.5 Empirical Analysis of the SwissAir Case

SwissAir, a subsidiary of SwissAir Group (SAirG), defaulted in October 2001. A threat of default of such a large contractor was the contagion effect on the network of SAirG suppliers. We apply the network pricing model to the analysis of loan yields of SAirG suppliers. Our sample contains a total of 56 SAirG suppliers. The set of suppliers consists of small and medium size enterprises in the canton of Zürich as well as some large, internationally active firms. All suppliers were also counterparties of a Swiss bank. Hence, it was a priori not clear to the bank whether the SwissAir Airlines default would contage the bank's counterparties. The SAirG network with all its subsidiaries and supplier firms considered is depicted in Figure 1.6. SAirG consists of four subsidiaries, SwissAir Airlines, SwissAir Services, SwissAir Relations and SwissAir Logistics, with SwissAir being a part of SwissAir Airlines. We treat all the subsidiaries as one firm in a network. The credit quality data provided by the bank includes the relationship of every supplier to the particular SAirG subsidiary. We assume that there are no business relationship between SAirG suppliers. The network is therefore star shaped with SAirG as its center.

We answer the following two questions. Do the bank's credit ratings of SAirG suppliers provide an indication for the potential default of SwissAir or did the suppliers itself anticipate it? How well does the network pricing model predict the credit ratings of SAirG suppliers after the SwissAir default?

[8] Collin-Dufresne, Goldstein, and Hugonnier (2004) shows possible mispricing of the CDO tranche yields if firm-firm dependence is not considered.

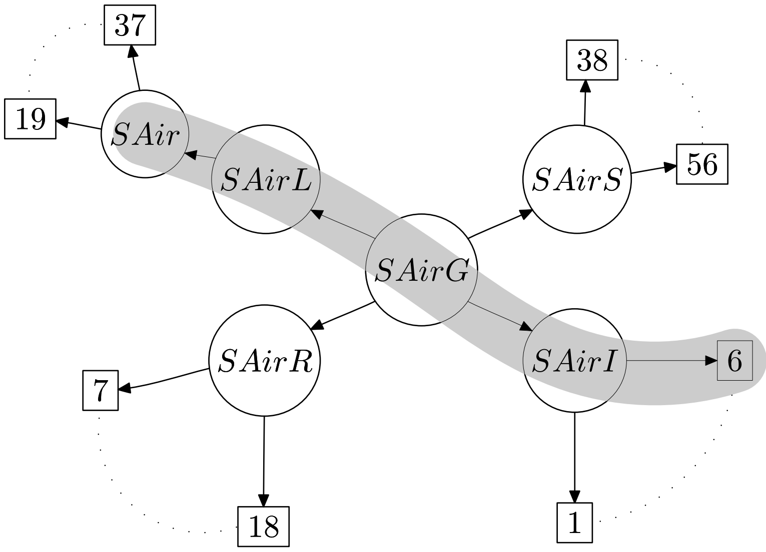


Figure 1.6: The network of SwissAir Group. SAirG represents the SwissAir Group, SAirL stands for SwissAir Airlines, SAirS for SwissAir Services, SAirR for SwissAir Relations and SAirI for SwissAir Logistics. The SAir denotes the SwissAir. The subsidiaries of SwissAir Group are shown in circles and the suppliers in rectangles. The suppliers of each subsidiary are denoted by numbers from 1 to 56. The grayed path indicates the contagion possibility from SAir to the supplier of another SAirG daughter company.

To answer the first question we denote by $Y_{i,t}$ the rating[9] in numerical terms (from 1 to 8) of firm i ($i = 1, \ldots, \bar{i} = 56$) at time t, $t = t_0, \ldots, t_1$, where $t_0 = D - 5$, $t_1 = D + 5$ and D denotes the default time of SwissAir (October 2001). The data provided are semi-annual. The measure for bank default anticipation we propose is $A = \frac{1}{\bar{i}} \sum_{i=1}^{\bar{i}} A_i$, where

$$A_i \quad = \quad \frac{\sum_{t=t_0}^{t_1} (Y_{i,t} - Y_{i,t-1})^2}{\left(\sum_{t=t_0}^{t_1} (Y_{i,t} - Y_{i,t-1})\right)^2} = \frac{\sum_{t=t_0}^{t_1} (Y_{i,t} - Y_{i,t-1})^2}{(Y_{i,t} - Y_{i,0})^2}$$

measures the bank's anticipation of the credit rating degradation of supplier i. The intuition for this measure is that for positive x_i ($i = 1, \ldots, N$) we have the inequality $\sum_{i=1}^{N} x_i^2 \leq \left(\sum_{i=1}^{N} x_i\right)^2$ with equality when all but one x_i are zero. Setting $x_{i,t} = Y_{i,t} - Y_{i,t-1}$, we have that $A_i \approx 1$ when all but one credit rating migrations are close to zero. This corresponds to a sudden drop in credit quality of firm i. Conversely if many x_{it} are nonzero, then $A_i \approx 0$, and the bank is likely to anticipate the degrading credit quality of the supplier i. The numerical value obtained for A from the data is $A_B = 0.66$. A Monte-Carlo simulation based on transitional S&P probabilities for the period 1981-1996, with the same proportion of firms corresponding to rating categories as in the SAirG sample, gives a value of $A_P = 0.72$ with variance 0.003. Since the banks's default anticipation value A_B is significantly lower than the statistically obtained A_P, we conclude that in the time leading towards SwissAir default a combination of the following events has occured: (a) The bank correctly gradually anticipated credit quality decrease of SAirG suppliers and (b) The SAirG suppliers themselves anticipated SwissAir default and diversified their business activities accordingly.

[9]The rating classes can be mapped into rating classes of standard credit quality assessment agencies, such as S&P and Moody's.

We consider next the empirical validity of the network rating model. We select in our primary sample those suppliers whose total business volume with SAirG was larger than 10%. The average business volume percentage over the entire sample is 33%. The credit history of each supplier exists for the last 5 years before the default and semi-annually for 4 years after the default of SwissAir. We assume that the credits were 3 year contracts, which corresponds to the average credit duration of bank's loans. Along with the credit quality of the suppliers, their leverage ratio, relative size in terms of SAirG, and the extent of business volume was provided. The amount of business volume proxies for the relative strength of business dependence between firms, i.e. the matrix $\boldsymbol{E}$ and the average intensity of buy orders proxies for $\underline{\lambda}$. We assume that the payments are on a monthly basis, i.e. $\underline{\lambda} = 12 \cdot \underline{1}$ and that the external cash flows of suppliers are as in the Merton model - the matrix $\boldsymbol{B}$ is diagonal, reflecting the market completeness assumption in Proposition 1.3.1. Additionally, we assume that there is no supplier-suppliers dependence. The only network dependency exists between the supplier and the SwissAir Group. Hence, the network is star shaped and the network dependency matrix $\boldsymbol{E}$ is of rank 1.

Since the pre-default leverage and business volume data are not available, we proceed as follows. We first estimate the level of external cash flows not related to SAirG for each supplier. For that purpose we use post-default data and estimate the volatility of external cash flows by equations (1.14)-(1.17) using the nonlinear regression method of Griliches and Intiligator (1983). We then add the network component to the model, see equation (1.17), thereby obtaining the volatility of cash flows by the suppliers before the SwissAir default, when SwissAir still constitued an amount of business volume by each supplier. Using again equation (1.14), we obtain the predicted yield of corporate debt.

Let y_t^i be the corporate debt yield of firm i at time t and let $d(L_i^t, U_i^t, B_i)$ be the computed yield from equation (1.17) with $\boldsymbol{F}$ given in (1.9) when the i-th firm's leverage at time t equals L_i^t, loan duration is U_i^t and B_i is the extent of external cash flows to the firm. The estimating equation then reads

$$y_t^i = d(L_i^t, U_i^t, B_i) + \varepsilon_t^i$$

where the error terms ε_t^i satisfies the same assumption as in Griliches and Intiligator (1983). The results are presented in Table 1.2. The second column presents the true debt yield of the supplier as provided by the bank for each of the 19 supplier firms with business volume exceeding[10] the 10% level. The third and fourth columns give the predicted corporate yield as computed from the network pricing and the Merton's model. The fifth and sixth columns show the relative yield error of the individual supplier for both models. The average relative yield error of the network model is 18% compared to the 89% average relative predicition error of the Merton model. Moreover, the Merton model underestimates corporate debt yield spreads in 13 out of 19 cases. This confirms the empirical results in Eom, Helwege, and Huang (2004).

The analysis of SAirG suppliers suggests that firms should not be considered in isolation contrary to standard models, such as KMV, CreditPlus and CreditMetrics. In the presence of a large contractor, an additional yield premium should be added *even though* the cash flows may not appear volatile. The additional yield premium is interpreted as a "buyer contagion default premium".

[10]This restriction is imposed in order to satisfy the normal market conditions, i.e. the network component dominates the external cash flow component, see Proposition 1.3.1.

Firm #	Bank yield	Model yield	Merton y.	RE (Model)	RE (Merton)
1	3.54%	4.04%	3.38%	13.99%	4.65%
2	3.81%	3.56%	18.96%	6.45%	397.59%
3	3.57%	2.18%	15.81%	38.95%	342.93%
4	3.54%	2.13%	2.49%	39.82%	29.71%
5	3.81%	3.81%	4.25%	0.06%	11.50%
6	3.54%	2.39%	2.62%	32.62%	25.97%
7	3.54%	2.57%	11.39%	27.32%	221.77%
8	3.54%	2.73%	2.80%	22.83%	20.94%
9	3.54%	2.80%	6.07%	20.85%	71.38%
10	3.54%	4.48%	3.86%	26.55%	9.10%
11	3.81%	3.47%	3.94%	8.88%	3.52%
12	3.81%	2.45%	2.58%	35.63%	32.30%
13	3.57%	3.40%	7.82%	4.78%	119.09%
14	3.81%	3.87%	5.30%	1.69%	39.12%
15	3.54%	3.41%	2.14%	3.79%	39.47%
16	3.81%	3.49%	5.85%	8.38%	53.49%
17	3.54%	2.13%	9.68%	39.83%	173.51%

Table 1.2: The consequitive columns represent: the firm number, bank (true) corporate yield, yield as predicted by the network model, yield predicted by the Merton (1974) model and the relative errors for both the network and the Merton model. In the sample are 19 firms with total business relationship percentage to SAirG exceeding 10%. The relative prediction error of both models is computed as $|\frac{\text{Bank yield}-\text{Model yield}}{\text{Bank yield}}|$.

1.6 Conclusions

This dissertation chapter lays out a theory of securities pricing in the presence of buyer-supplier networks, emphasizing the effect on the structure of financial markets. We show that the network model has the potential to reconcile over- and underestimation issues of structural models reported in empirical literature by inducing a network dependence of firms to their buyers and suppliers. The degree of network market incompleteness is determined only by the topology of the network. More precisely, the dimension of the space of equivalent martingale measures increases with every buyer-supplier chain. When the buy orders occur rapidly we obtain a limit to the classical Brownain financial market model.

We identify the risk structure of corporate securities by connecting them to market prices of buy-orders. Corporate debt yield spreads can increase due to two manifestly different reasons: either by exposure to direct business relationships of the firm to its buyers or by increase in market prices of buy orders which are determined globally - the model induces contagion effects and ties the systemic risk of the economy to the prices of individual securities. We prove that the prices of firm's obligations depend not only on its immediate buyers but on the whole buy-supply chain.

In a typical situation the model predicts an increase in corporate and portfolio excess yields with increasing network dependency of firms and a higher slope of excess yield curve (with respect to network dependency) for firms lower in the buyer-supplier chains.

The model is empirically tested on the network of subcontractors of the SwissAir Group. The default of SwissAir had the potential to induce a contagion effect in its subcontractors network. We show that the subcontractors diversified their business activities and anticipated the financial distress of SwissAir. The network model has an average 18% yield spread relative prediction error compared to an 89% error for the Merton model.

The network model sheds light on the asset risk structure of vertically connected firms and can be further improved by considering endogenous network formation.

1.A Appendix

1.A.1 Proofs of Section 1.2

Proof. (of Proposition 1.2.1.) The result follows by simple computation

$$\begin{aligned}
I_i(t) &= \mathbb{E}_t^{\mathbb{Q}}\left(\int_t^\infty e^{-r(s-t)}A_i(s)\left(\sum_{j=1}^N E_{ij}P_i\,\mathrm{d}N_j(s)+B_iP_i\,\mathrm{d}N_i(s)+\eta_i\,\mathrm{d}s\right)\right)\\
&= \int_t^\infty e^{-r(s-t)}\mathbb{E}_t^{\mathbb{Q}}\left[A_i(s)\mathbb{E}_s^{\mathbb{Q}}\left(\sum_{j=1}^N E_{ij}P_i\,\mathrm{d}N_j(s)+B_iP_i\,\mathrm{d}N_i(s)+\eta_i\,\mathrm{d}s\right)\right]\\
&= \int_t^\infty e^{-r(s-t)}\mathbb{E}_t^{\mathbb{Q}}\left[A_i(s)\left(\sum_{j=1}^N E_{ij}P_i\gamma_j\,\mathrm{d}s+B_iP_i\gamma_i\,\mathrm{d}s+\eta_i\,\mathrm{d}s\right)\right]\\
&= \int_t^\infty e^{-r(s-t)}A_i(t)\exp\left(\sum_{j=1}^N(E_{ij}P_i\gamma_j(s-t)+B_iP_i\gamma_i(s-t)+\eta_i(s-t)\right)\\
&\qquad\qquad\left[\sum_{j=1}^N E_{ij}P_i\gamma_j+B_iP_i\gamma_i+\eta_i\right]\mathrm{d}s\\
&= \frac{\sum_{j=1}^N E_{ij}P_i\gamma_j+B_iP_i\gamma_i+\eta_i}{r-\left(\sum_{j=1}^N E_{ij}P_i\gamma_j+B_iP_i\gamma_i+\eta_i\right)}A_i.
\end{aligned}$$

This proves that $\frac{\mathrm{d}I_i}{I_i}=\frac{\mathrm{d}A_i}{A_i}$ and

$$I_i(0)=A_i(0)\frac{\sum_{j=1}^N E_{ij}P_i\gamma_j+B_iP_i\gamma_i+\eta_i}{r-\left(\sum_{j=1}^N E_{ij}P_i\gamma_j+B_iP_i\gamma_i+\eta_i\right)}=A_i(0)K_i. \tag{1.18}$$

□

Proof. (of Proposition 1.2.5) Let $\underline{M}(t)=\underline{N}(t)-\underline{\lambda}t$ be the compensated martingale associated with $\underline{N}$. We set $\underline{Y}=\log(\underline{A})$ and compute

$$\begin{aligned}
\underline{Y}(t) &= \underline{Y}(0)+\underline{\eta}t+\tilde{\boldsymbol{E}}\underline{M}(t)+\tilde{\boldsymbol{E}}\underline{\lambda}t\\
&= \underline{Y}(0)+\underline{\eta}t+\tilde{\boldsymbol{E}}\boldsymbol{\lambda}^{1/2}\boldsymbol{\lambda}^{-1/2}\underline{M}(t)+\tilde{\boldsymbol{E}}\underline{\lambda}t
\end{aligned}$$

Donsker's theorem gives us that the process $\underline{S}_n(t)=\frac{1}{\sqrt{n}}\boldsymbol{\lambda}^{-1/2}\underline{M}(nt)$ converges in distribution (as $n\to\infty$) as a process to the N dimensional Brownian motion $\underline{W}$. By assumptions of the Proposition and the continuous mapping theorem we get the desired result.

It remains to prove the last part of the Proposition. Before the default of firm i we have (by the first condition (1.9)) to hold

$$\mu_i=\lim_{n\to\infty}\sqrt{n}\left(\eta_i^n+\sum_{j=1}^N\tilde{E}_{ij}^n\lambda_j^n\right).$$

After the default of some firm $j \neq i$ the same condition has to hold but in a smaller network, i.e.

$$\mu_i = \lim_{n\to\infty} \sqrt{n} \left(\eta_i^n + \sum_{j \neq i} \tilde{E}_{ij}^n \lambda_j^n \right).$$

Subtracting both condition we get that

$$0 = \lim_{n\to\infty} \sqrt{n} E_{ij}^n \lambda_j^n.$$

This is precisely the condition (1.11). $\square$

1.A.2 Proofs of Section 1.3

Proof. (of Proposition 1.3.2) Let the network of firms $\mathcal{G} = \cup_{l=1}^R P_l$ where P_l are disjoint paths of lengths d_l in $\mathcal{G}$ ordered by maximum length, i.e. $d_1 \geq d_2 \geq \ldots \geq d_R$. We will prove that the Jordan cannonical form $\boldsymbol{J_E}$ of $\boldsymbol{E}$ can be written as

$$\boldsymbol{J_E} = \begin{bmatrix} \boldsymbol{J}_1 & & & \\ & \boldsymbol{J}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{J}_R \end{bmatrix}, \tag{1.19}$$

where

$$\boldsymbol{J}_l = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in \mathbb{R}^{d_l \times d_l},$$

that is the Jordan cages are the same length as the paths in the decomposition of $\mathcal{G}$. This proves the Proposition.

Since $\mathcal{G}$ is a buyer supplier chain, the adjacency matrix allows for the reordering of firms, such that $\boldsymbol{E}'$ is upper triangular with zeros on the main diagonal. The Jordan canonical form of such a matrix is as in (1.19) - matrices $\boldsymbol{E}$ and $\boldsymbol{E}'$ have the same Jordan form. What remains to be shown is that the Jordan cages $\boldsymbol{J}_1, \ldots, \boldsymbol{J}_R$ correspond to the disjunct paths ordered by maximum length. Let us assume that the Jordan cages in (1.19) are also ordered by size, i.e. size $\boldsymbol{J}_1 \geq$ size $\boldsymbol{J}_2 \geq \ldots \geq$ size $\boldsymbol{J}_R$.

We use the following two transformations on the matrix $\boldsymbol{E}'$ which both preserve the Jordan canonical form: (1) $L_{ij}(\alpha)$ adds to column i the α-multiple of column j as well as to row j the $-\alpha$-multiple of row i (see Bujosa, Criado, and Vega (1998) for elaboration) and (2) S_{ij} swaps i-th and j-th row and column in $\boldsymbol{E}$. By applying the series of transformations in Lemma 1.A.1. This reduces the matrix to the classical Jordan form. $\square$

Lemma 1.A.1. *Let $\boldsymbol{A}$ be a strictly upper triangular matrix with strictly positive values on the first upper-diagonal. Then $\boldsymbol{A}$ has only one Jordan cage.*

Proof. It is easily seen that $\underline{e}_1$ (the first basis vector) is in the kernel of $\boldsymbol{A}$. We will show that there does not exist any other vector in the kernel. This will prove that there is only one Jordan cage. Let us assume that another $\underline{v} = \lambda_2 \underline{e}_2 + \ldots + \lambda_n \underline{e}_n$ is in the kernel of $\boldsymbol{A}$ (we can assume that $\underline{v} \perp \underline{e}_1$) where not all $\lambda_i = 0$. Then

$$\begin{aligned}\boldsymbol{A}\underline{v} &= \lambda_2 a_{12} \underline{e}_1 + \\ &\quad + \lambda_3 a_{13} \underline{e}_1 + \lambda_3 a_{23} \underline{e}_2 \\ &\quad + \ldots\end{aligned}$$

In order for $\boldsymbol{A}\underline{v} = \underline{0}$ it necessary needs to hold $\lambda_i = 0$ for all i ($a_{ij} \geq 0$ and $a_{i,i+1} > 0$) which contradicts the assumption.

Moreover, the sequences of transformations to do this are ones that at all times preserve the upper-triangular structure of the matrix. We first multiply the matrix $\boldsymbol{A}$ by the $\boldsymbol{P} = \mathrm{diag}(p_1, \ldots, p_n)$ such that

$$\begin{aligned} p_1^{-1} a_{12} p_2 &= 1 \\ p_2^{-1} a_{23} p_3 &= 1 \\ &\vdots \\ p_{n-1}^{-1} a_{n-1,n} p_n &= 1 \end{aligned}$$

We then apply the following sequence of transformations $L_{n,n-1}(-a_{n-2,n}), L_{-a_{n-3,n}}, \ldots, L_{n-1,2}(-a_{1,n})$. This makes the matrix $\boldsymbol{A}$ with zeroes in the last column except for 1 in the upper-diagonal. We repeat the following sequence until we obtain the Jordan form. $\square$

Lemma 1.A.2. *Suppose that the firm network graph can be partitioned into K connected components. Then the rank of $\boldsymbol{E}$ is the sum of ranks of adjacency matrices of the connected components. Specifically rank $\boldsymbol{E} \leq N - K$.*

Proof. If there are K connected components in a graph, then $\boldsymbol{E}$ can be decomposed as

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_1 & & & \\ & \boldsymbol{E}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{E}_K \end{bmatrix},$$

where $\boldsymbol{E}_k$ is the adjacency matrix of the k-th connected component. Then rank $(\boldsymbol{E}) = \sum_{k=1}^{K}$ rank $(\boldsymbol{E}_k) \leq N - K$, since rank $(\boldsymbol{E}_k) \leq$ dim $(\boldsymbol{E}_k) - 1$. $\square$

The following proposition is a restatement of Theorem 2 of He, Keirstead, and Rebholz (1998) and the notation is adopted from there. τ_i denotes the first hitting time of the linear Brownian motion process X_i of D_i, $D_i < 0$.

Proposition 1.A.3. *Let $X_i = \alpha_i t + \sigma_i W_i$ ($i = 1, 2$), where W_1 and W_2 are two correlated Brownian motions with correlation ρ and running minimum m_i. Then the probability density*

$$\mathbb{P}(X_1(t) \in dx_1, X_2(t) \in dx_2, m_1(t) \geq m_1, m_2(t) \geq m_2) = p(x_1, x_2, t; m_1, m_2) \tag{1.20}$$

is

$$p(x_1, x_2, t; m_1, m_2) = \frac{e^{a_1 x_1 + a_2 x_2 + bt}}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} h(x_1, x_2, t; m_1, m_2)$$

if $x_1 \geq m_1$*,* $x_2 \geq m_2$*,* $m_1, m_2 < 0$ *and zero otherwise and*

$$h(x_1, x_2, t; m_1, m_2) = \frac{2}{\beta t} \sum_{n=1}^{\infty} e^{-\frac{r^2+r_0^2}{2t}} \sin\left(\frac{n\pi\theta_0}{\beta}\right) \sin\left(\frac{n\pi\theta}{\beta}\right) I_{(n\pi)/\beta}\left(\frac{r r_0}{t}\right)$$

where

$$\begin{aligned}
a_1 &= \frac{\alpha_1\sigma_2 - \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2} \\
a_2 &= \frac{\alpha_2\sigma_1 - \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2} \\
b &= -\alpha_1 a_1 - \alpha_2 a_2 + \frac{1}{2}(\sigma_1^2 a_1^2 + \sigma_2^2 a_2^2) + \rho\sigma_1\sigma_2 a_1 a_2 \\
\tan\beta &= -\frac{\sqrt{1-\rho^2}}{\rho} \qquad \beta \in [0, \pi] \\
z_1 &= \frac{1}{\sqrt{1-\rho^2}}\left[\frac{x_1 - m_1}{\sigma_1} - \rho\frac{x_2 - m_2}{\sigma_2}\right] \\
z_2 &= \frac{x_2 - m_2}{\sigma_2} \\
z_{10} &= \frac{1}{\sqrt{1-\rho^2}}\left[-\frac{m_1}{\sigma_1} + \frac{\rho m_2}{\sigma_2}\right] \\
z_{20} &= -\frac{m_2}{\sigma_2} \\
r &= \sqrt{z_1^2 + z_2^2} \\
\tan\theta &= \frac{z_2}{z_1} \qquad \theta \in [0, \beta] \\
r_0 &= \sqrt{z_{10}^2 + z_{20}^2} \\
\tan\theta_0 &= \frac{z_{20}}{z_{10}} \qquad \theta_0 \in [0, \beta]
\end{aligned}$$

and I_ζ *is the modified Bessel function.*

Proposition 1.A.4. *The following identity holds*

$$\mathbb{P}(\tau_1 \in [t, t + \mathrm{d}t), X_2(t) \in \mathrm{d}x_2, m_2(t) \geq D_2) = p(-D_1, x_2, t; D_1, D_2)\,\mathrm{d}x_2\ \mathrm{d}t$$

where p *is defined as in (1.20) in Proposition 1.A.3.*

Proof. We compute

$$\begin{aligned}
&\mathbb{P}\left(\tau_1 \in [t, t+dt), X_2(t) \in \mathrm{d}x_2, \underline{X}_2(t) \geq D_2\right) \\
&\quad = \mathbb{P}(X_1(t) \in [D_1, D_1 - \mathrm{d}D_1), \underline{X}_1(t) \geq D_1, X_2(t) \in \mathrm{d}x_2, \underline{X}_2(t) \geq D_2) \\
&\quad = p(-D_1, x_2, t; D_1, D_2)\,\mathrm{d}x_2\ \mathrm{d}t
\end{aligned}$$

by the reflection property of the first Brownian motion X_1. ☐

We use the tower property of conditional expectation to compute the expected excess survival time of the supplier firm 2 given that it defaults after firm 1 is

$$\begin{aligned}\mathbb{E}(\tau_2-\tau_1|\tau_2>\tau_1) &= \int_0^\infty \mathrm{d}u \int_{D_2}^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}t\, \mathbb{P}(\tau_2-\tau_1>u|X_2(t)=x_2)\cdot \\ &\qquad \mathbb{P}(\tau_1\in[t,t+\mathrm{d}t), X_2(t)\in \mathrm{d}x_2, m_2(t)\geq D_2) \\ &= \int_0^\infty \mathrm{d}u \int_{D_2}^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}t\, \mathbb{P}(X_2^c(u)\geq D_2|X_2^c(0)=x_2)\cdot \\ &\qquad \mathbb{P}(\tau_1\in[t,t+\mathrm{d}t), X_2(t)\in \mathrm{d}x_2, m_2(t)\geq D_2),\end{aligned}$$

since the process X_2^c without the firm 1 is again a time homogenous Brownian motion with drift.

1.A.3 Proofs of Section 1.4

Proof. (of Proposition 1.4.1.) Let there be a directed path between $p_1=a\to p_2\to\ldots\to p_k=b$. We show that then $\gamma_b=\gamma_b(B_a,\eta_a,E_{ak})$ which in turn proves the statement in the proposition. We prove this by induction on the length k of the path between a and b. The basis for induction $k=1$ is the case where there is a direct connection $a\to b$. Writing equation (1.12) for γ_b we have

$$P_bB_b\gamma_b+\sum_{i\neq a}E_{bi}P_b\gamma_i+E_{ba}P_b\gamma_a=\delta_b-\eta_b.$$

Since $\gamma_a=\gamma_a(B_a,\eta_a,E_{ak})$ and the statement follows. We have proven the basis of induction. We now assume that we have the proven the dependence for the paths of length k. The statement then follows by similar considerations as the one in (1.21): $\gamma_{k+1}=\gamma_{k+1}(\gamma_k)$ which implies the result. □

Proof. (of Proposition 1.4.2.) We compute

$$\begin{aligned}\gamma_i-\gamma_{i+1} &= \frac{\delta_i-\eta_i}{P_iB_i}-\frac{\delta_{i+1}-\eta_{i+1}}{P_{i+1}B_{i+1}}-\frac{\sum_{j=i+1}^N E_{ij}\gamma_j}{B_i}+\frac{\sum_{j=i+2}^N E_{i+1,j}\gamma_j}{B_{i+1}} \\ &= \underbrace{\frac{\delta_i-\eta_i}{P_iB_i}-\frac{\delta_{i+1}-\eta_{i+1}}{P_{i+1}B_{i+1}}}_{\geq 0}-\underbrace{\frac{E_{i,i+1}\gamma_{i+1}}{B_i}}_{\geq 0}-\underbrace{\sum_{j=i+2}^N\left(\frac{E_{i,j}}{B_i}-\frac{E_{i+1,j}}{B_{i+1}}\right)\gamma_j}_{\geq 0} \\ &\geq 0,\end{aligned}$$

which together with the assumptions proves the result. □

Proof. (of equation 1.13.) The equation follows from the following relationship:

$$\begin{aligned}\mu_i\,\mathrm{d}t &= \mathrm{d}\mathbb{P}(A_i(t+\mathrm{d}t)\leq\gamma_i|A_i(t),A_i(t)>\gamma_i) \\ &= \mathbb{P}\left(\bigcup_j A_i(t)\leq\gamma_i(1+E_{ij}),\,\mathrm{d}N_j=1\right) \\ &= \sum_j \mathbb{P}(\mathrm{d}N_j=1)\cdot\mathbb{1}(E_{ij}<0)\cdot\mathbb{1}(A_i(t)\leq\gamma_i(1+E_{ij}))\end{aligned}$$

The remaining part of the equation (1.13) follows from the facts that $\mathbb{P}(\mathrm{d}N_j=1)=\lambda_j\,\mathrm{d}t$, $E_{ii}=B_i$ and taking into account only the firms in a network that are still active. □

Proof. (of Proposition 1.4.3.) The price of corporate debt in this case is

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(e^{-rT}\min(A_i(T),D_i)) &= \mathbb{E}^{\mathbb{Q}}(e^{-rT}A_i(T)\mathbb{1}(A_i(T)\le D_i)+e^{-rT}D_i\mathbb{1}(A_i(T)>D_i))\\ &= e^{-rT}D_i\mathbb{Q}(A_i(T)\ge D_i)+\mathbb{E}^{\mathbb{Q}}(e^{-rT}A_i(T)\mathbb{1}(A_i(T)\le D_i)).\end{aligned}$$

We note that under the measure $\mathbb{Q}$ we have that the process A_i at time t is given by

$$A_i(t) = A_i(0)\exp\left(\eta_i t+\sum_{j=1}^{N}\tilde{E}_{ij}N_j(t)\right)$$

where N_j is a Poisson process of constant intensity λ_j. Therefore

$$\begin{aligned}\mathbb{Q}(A_i(T)\ge D_i) &= \mathbb{Q}\left(\sum_{j=1}^{N}\tilde{E}_{ij}N_j(T)\ge\log\frac{D_i}{A_i(0)}-\eta_i T\right)\\ &= 1-N^{\lambda_1,\lambda_2,\ldots,\lambda_N}_{\tilde{E}_{i1},\tilde{E}_{i2},\ldots,\tilde{E}_{i,N}}(d)\end{aligned}$$

where

$$d=\log\frac{D_i}{A_i(0)}-\eta_i T.$$

To handle the second integral we make the change of measure

$$\frac{\mathrm{d}\mathbb{O}}{\mathrm{d}\mathbb{Q}}=\exp\left(\sum_{j=1}^{N}\tilde{E}_{ij}N_j(T)-E_{ij}P_i\lambda_j T\right).$$

Under this change of measure the processes

$$N_j(t)-(1+E_{ij}P_i)\lambda_j t$$

are local martingales which by the martingale classification theorem shows that N_j is a Poisson process under $\mathbb{O}$ with intensity $(1+E_{ij}P_i)\lambda_j$. Therefore

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(e^{-rT}A_i(T)\cdot\mathbb{1}(A_i(T)\le D_i)) &= A_i(0)e^{\left(\sum_{j=1}^{N}E_{ij}P_i\lambda_j-r\right)T}\mathbb{O}(A_i(T)\le D_i)\\ &= A_i(0)e^{\left(\sum_{j=1}^{N}E_{ij}P_i\lambda_j-r\right)T}\cdot\\ &\qquad N^{(1+E_{i1}P_i)\lambda_1,(1+E_{i2}P_i)\lambda_2,\ldots,(1+E_{iN}P_i)\lambda_N}_{\tilde{E}_{i1},\tilde{E}_{i2},\ldots,\tilde{E}_{i,N}}(d).\end{aligned}$$

The probability

$$N^{\lambda_1,\ldots,\lambda_n}_{E_1,\ldots,E_n}(x)=\mathbb{P}\left(\sum_{i=1}^{n}E_iN_i(T)\le x\right)$$

where N_i are independent Poisson processes of intensity λ_i can be computed from the characteristic function φ_L of $L=\sum_{i=1}^{n}E_iN_i(T)$ which equals

$$\varphi_L(y)=\prod_{j=1}^{n}\exp\left(\lambda_j T(e^{iE_jy}-1)\right).$$

We then numerically compute the

$$N^{\lambda_1,\dots,\lambda_n}_{E_1,\dots,E_n}(x_1) - N^{\lambda_1,\dots,\lambda_n}_{E_1,\dots,E_n}(x_2) = \lim_{\tau\to\infty}\frac{1}{2\pi}\int_{-\tau}^{\tau}\frac{e^{-iyx_2}-e^{-iyx_1}}{iy}\varphi_L(y)\,\mathrm{d}y.$$

To develop the hedging formulas we start with

$$\begin{aligned}\mathrm{d}V &= \sum_{i=1}^{N}\phi_i\,\mathrm{d}A_i\\ &= \sum_{i=1}^{N}\phi_i A_i\left(\sum_{j=1}^{N}E_{ij}P_i\,\mathrm{d}N_j\right)\end{aligned}$$

Using the Ito-formula for Poisson processes gets us

$$\begin{aligned}\mathrm{d}V(t,A_1,\dots,A_N) &= V_t\,\mathrm{d}t + \sum_{i=1}^{N}V_{A_i}\,\mathrm{d}A_i\\ &\quad+\sum_{k=1}^{N}\big(V(t,A_1(1+E_{1k}P_1),A_2(1+E_{2k}P_2),\dots,A_N(1+E_{Nk}P_N))\\ &\qquad -V(t,A_1,A_2,\dots,A_N)-\sum_{j=1}^{N}V_{A_i}A_iE_{ik}P_i\Big)\,\mathrm{d}N_k \end{aligned}\tag{1.21}$$

Rewriting the equation (1.21) we get

$$\begin{aligned}\mathrm{d}V(t,A_1,\dots,A_N) &= V_t\,\mathrm{d}t + \sum_{k=1}^{N}\mathrm{d}N_k\sum_{i=1}^{N}V_{A_i}A_iE_{ik}P_i\\ &\quad+\sum_{k=1}^{N}\big(V(t,A_1(1+E_{1k}P_1),A_2(1+E_{2k}P_2),\dots,A_N(1+E_{Nk}P_N))\\ &\qquad -V(t,A_1,A_2,\dots,A_N)-\sum_{j=1}^{N}V_{A_i}A_iE_{ik}P_i\Big)\,\mathrm{d}N_k. \end{aligned}\tag{1.22}$$

Solving equation (1.1) for $\mathrm{d}\underline{N}$ gives us

$$\mathrm{d}\underline{N} = \boldsymbol{P}^{-1}\boldsymbol{E}^{-1}\left(\boldsymbol{A}^{-1}\,\mathrm{d}\underline{A} - \underline{\eta}\,\mathrm{d}t\right).$$

Inserting the latter expression into (1.22) we arrive at

$$\begin{aligned}\mathrm{d}V(t,A_1,\dots,A_N) &= \dots\,\mathrm{d}t + \sum_{k=1}^{N}\big(V(t,A_1(1+E_{1k}P_1),A_2(1+E_{2k}P_2),\dots,A_N(1+E_{Nk}P_N))\\ &\qquad -V(t,A_1,A_2,\dots,A_N)\big)\,\mathrm{d}N_k\end{aligned}$$

Let us denote by $\underline{U} = (U_1,\dots,U_N)'$ where

$$U_k = V(t,A_1(1+E_{1k}P_1),A_2(1+E_{2k}P_2),\dots,A_N(1+E_{Nk}P_N)) - V(t,A_1,A_2,\dots,A_N).$$

The weights in the hedging portfolio are then

$$\underline{U}'(\boldsymbol{AEP})^{-1}.$$

□

The following proposition is immediately apparent from Proposition 1.4.3.

Proposition 1.A.5. *If the network $\mathcal{G}$ can be decomposed into several connected components, then the securities' prices depend only on the parameters of the connected components to which they belong.*

Lemma 1.A.6. *If the protection buyer pays a constant amount U to the protection seller at time intervals $\{t_i\}_i$, which pays D to the protection buyer in the event of a default of the referenced entity. Therefore the fair price for the protection seller charging the protection buyer can be expressed as*

$$\frac{U}{D} = \frac{\mathbb{E}^{\mathbb{Q}}[e^{-r\tilde{\tau}_i}1(\tilde{\tau}_i < T)]}{\sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(t_j < \tilde{\tau}_i)}. \tag{1.23}$$

Proof. (of lemma 1.A.6.) We start from

$$\frac{U}{D} = \frac{\mathbb{E}^{\mathbb{Q}}[e^{-r\tilde{\tau}_i}1(\tilde{\tau}_i < T)]}{\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{T\wedge\tilde{\tau}_i} e^{-rt_i}\right]}. \tag{1.24}$$

The formula in the denominator can be simplified to get

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{T\wedge\tilde{\tau}_i} e^{-rt_i}\right] &= \sum_{n=0}^{T}\sum_{j=1}^{n} e^{-rt_j}\mathbb{Q}(t_n \leq \tilde{\tau}_i < t_{n+1}) + \sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(\tilde{\tau}_i > T) \\
&= \sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(t_j \leq \tilde{\tau}_i < t_N)q + \sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(\tilde{\tau}_i \geq t_N) \\
&= \sum_{j=1}^{T} e^{-rt_j}\mathbb{Q}(t_j < \tilde{\tau}_i),
\end{aligned}$$

whereas the numerator reduces to the truncated Laplace transform[11] in a network. □

Proof. (of Proposition 1.4.5.) Equation (1.23) involves computation of the truncated Laplace transform. In the case when default can occur only at maturity, explicit formulas for truncated Laplace transform have been developed (see Guha and Sbuelz (2003), page 10). In our case we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[e^{-r\tilde{\tau}_i}1(\tilde{\tau}_i < T)] &= \exp\left(-\frac{x_i(\alpha_i + \vartheta_i)}{\sigma_i^2}\right) N\left(\frac{-x_i + \vartheta_i T}{\sigma_i\sqrt{T}}\right) \\
&\quad + \exp\left(-\frac{x_i(\alpha_i - \vartheta_i)}{\sigma_i^2}\right) N\left(\frac{-x_i - \vartheta_i T}{\sigma_i\sqrt{T}}\right)
\end{aligned} \tag{1.25}$$

where quantities are given as in the Proposition. □

[11]Related work on truncated Laplace transform was done by Campi and Sbuelz (2005) which also develop closed form expressions for the truncated Laplace transform in some specific cases of processes.

Proof. (of Proposition 1.4.6.) For the determination of the CDO prices we need to compute the distribution of the sum of correlated log-normal random variables. We use the approximation method od Fenton (1960). For the derivation we refer to the paper by Safak and Safak (1994) and the references therein. Fenton (1960) approximates the sum of log-normal distributions $\sum_{i=1}^{N}\exp(Y_i)$, where Y_i is normally distributed by mean m_i and variance σ_i^2 by another log-normal distribution $\exp(S_N)$, where S_N is normal with mean m_{S_N} and variance $\sigma_{S_N}^2$, by fitting the first two moments of the distribution to that of the sum. In our case the mean and the variance of the logarithm of individual summand is $m_i = \log(A_i(0)) + rT - \frac{1}{2}\sum_{j=1}^{N} F_{ij}^2 T$ and $\sigma_i^2 = \sum_{j=1}^{N} F_{ij}^2 T$ respectively. The correlation between Y_i and Y_j in the sum is $\rho_{ij} = \frac{\sum_{k=1}^{N} F_{ik}F_{jk}T}{\sigma_i\sigma_j}$. Fenton's method then gives us

$$\begin{aligned} m_{S_N} &= -\frac{1}{2}\sigma_{S_N}^2 + \log\left(\sum_{i=1}^{N}\exp\left(m_i + \frac{1}{2}\sigma_i^2\right)\right) \\ \sigma_{S_N}^2 &= \log\left(1 + \frac{\sum_{i=1}^{N}(e^{\sigma_i^2}-1)e^{2m_i+\sigma_i^2} + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} e^{m_i+\frac{1}{2}\sigma_i^2}e^{m_j+\frac{1}{2}\sigma_j^2}(e^{\rho_{ij}\sigma_i\sigma_j}-1)}{\left(\sum_{i=1}^{N} e^{m_i+\frac{1}{2}\sigma_i^2}\right)^2}\right) \end{aligned}$$

We define $d_k^1 = \frac{m_{S_N} - \log C_k}{\sigma_{S_N}}$ and similarly $d_k^2 = \frac{m_{S_N} + \sigma_{S_N}^2 - \log C_k}{\sigma_{S_N}}$. Then

$$\begin{aligned} \mathbb{Q}(\tilde{A} \geq C_k) &= \mathbb{Q}\left(m_{S_N} + \sigma_{S_N}\frac{W(T)}{\sqrt{T}} \geq \log C_k\right) \\ &= \mathbb{Q}\left(-\frac{W(T)}{\sqrt{T}} \leq \frac{m_{S_N} - \log C_k}{\sigma_{S_N}}\right) \\ &= N(d_k^1). \end{aligned}$$

Since $\tilde{A} = e^{m_{S_N} + \frac{\sigma_{S_N}}{\sqrt{T}}W(T)}$ we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\tilde{A}\chi(C_k \leq \tilde{A} \leq C_{k+1})] &= e^{m_{S_N}+\frac{1}{2}\sigma_{S_N}^2}\mathbb{E}^{\mathbb{Q}}\left[e^{-\frac{1}{2}\sigma_{S_N}^2 + \frac{\sigma_{S_N}}{\sqrt{T}}W(T)}\chi(C_k \leq \tilde{A} \leq C_{k+1})\right] \\ &= e^{m_{S_N}+\frac{1}{2}\sigma_{S_N}^2}\left(\mathbb{Q}^*(\tilde{A}^* \geq C_k) - \mathbb{Q}^*(\tilde{A}^* \geq C_{k+1})\right) \\ &= e^{m_{S_N}+\frac{1}{2}\sigma_{S_N}^2}\left(N(d_k^2) - N(d_{k+1}^2)\right), \end{aligned}$$

where $\tilde{A}^* = \exp(m_{S_N} + \sigma_{S_N}^2 + \frac{\sigma_{S_N}}{\sqrt{T}}\tilde{W}(T))$, $\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = e^{-\frac{1}{2}\sigma_{S_N}^2 + \frac{\sigma_{S_N}}{\sqrt{T}}W(T)}$ and $\tilde{W}$ is the Brownian motion under $\mathbb{Q}^*$. Gathering all terms together we get

$$\begin{aligned} V_k &= e^{-rT}F_k\mathbb{Q}(\tilde{A} \geq C_{k+1}) + e^{-rT}\mathbb{E}^{\mathbb{Q}}[(\tilde{A} - C_k)\chi(C_k \leq \tilde{A} \leq C_{k+1})] \\ &= e^{-rT}F_k\mathbb{Q}(\tilde{A} \geq C_{k+1}) - C_k e^{-rT}(\mathbb{Q}(\tilde{A} \geq C_k) - \mathbb{Q}(\tilde{A} \geq C_{k+1})) + \mathbb{E}^{\mathbb{Q}}[\tilde{A}\chi(C_k \leq \tilde{A} \leq C_{k+1})] \\ &= e^{-rT}F_k N(d_{k+1}^1) - C_k e^{-rT}(N(d_k^1) - N(d_{k+1}^1)) + e^{-rT+m_{S_N}+\frac{1}{2}\sigma_{S_N}^2}\left(N(d_k^2) - N(d_{k+1}^2)\right) \end{aligned}$$

□

1.B Additional Tables

$\boldsymbol{E}$	$\underline{P}$	$\underline{\lambda}$	$\boldsymbol{B}$	$\underline{D}$	$\underline{A}_0$	T
$\begin{bmatrix} 0 & 30 & 20 \\ 0 & 0 & 60 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.003 \\ 0.003 \\ 0.002 \end{bmatrix}$	$\begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix}$	$\begin{bmatrix} -10.000 & 0.000 & 0.000 \\ 0.000 & -8.000 & 0.000 \\ 0.000 & 0.000 & -7.000 \end{bmatrix}$	$\begin{bmatrix} 80 \\ 80 \\ 80 \end{bmatrix}$	$\begin{bmatrix} 100 \\ 110 \\ 130 \end{bmatrix}$	1

Table 1.3: Model parameters for the network of three firms. The entries in $\boldsymbol{E}$ reflect the relative business dependence between firms, i.e. the cash payments between firm 1 and 2 are large than between firms 1 and 3. The factor of proportionality $\boldsymbol{P}$ for payment streams is small and the average number of payments $\underline{\lambda}$ between firms high - a high transaction/low inter-firm payment economy. The volatility of external cash flows for the given parameters ranges between 0.1 to 0.2 for all firms in the network. We assume that there is no correlation between the external cash flows of the individual firms, i.e. $\boldsymbol{B}$ is diagonal. Hence, there is only network dependence between them. Time interval is normalized to 1.

Chapter 2

Mergers and Asset Prices in a Firm Network Economy

2.1 Introduction

The size and the scope of firms has long puzzled economic theory. In the classic general equilibrium setting, where firms are motivated by profit generation, only a single firm should exist. A general theory of mergers and acquisitions has not been properly accounted for in finance sholarship even though mergers have been much investigated. The motivation for merger waves is even included in the Brealey and Myers (2000) list of the ten unsolved questions of modern financial theory.

In this paper we examine the outcome of an economy populated by multiple firms facing risky profit streams and merger possibilities. By merging the firms create an *internal capital market* and shield themselves from the adverse movements in stochastic cash flows. As pointed out by Williamson (1987), the internalization of capital markets is an important part of a firm's strategy and the functioning of markets in general. The impact of firm's risk structure on the creation of internal capital markets is emphasized in Elgers and Clark (1980), p. 66 who state that

> "From a shareholder's standpoint, business combinations are justified when the market value of the equity shares of buyer and seller firms increases as a result of their intention to merge. The incremental value might accrue from expectations of the replacement of incompetent management, scale economies, extension of the product line, improved market control, *reduction of business risk*, or *changes in the financial structure*." [emphasis mine]

For example, the Neue Zürcher Zeitung (2007) report on News Corporation indicates this phenomena: Although the earnings of the film branch of 20th Century Fox suffered in 2007 the gains in Fox TV adequately offset them. In this case the aggregate volatility is reduced when firms conglomerate. The merger between YouTube and Google in late 2006, the attempt of EUREX to merge with LSE and the takeover of Wachovia by Wells Fargo further exemplify this tendency in conglomeration theory. Merger synergies, market concentration effects, coalitional externalities, firm growth potential, tax advantages or resource transfers are other factors that determine merger conditions.

In this paper we broadly define the term *coalition* as the cooperation of firms in forming internal capital markets. While classical mergers, mergers of equals, acquisitions or hostile takeovers are examples of coalition formation, cooperations and alliances are not. The following two questions will thus be addressed:

1. How does merger behavior depend on the risk structure of firms' cash flows, i.e. how does cash flow dependence between buyers and suppliers and firms' leverage ratios effect merger activity?

2. What are the post-merger share prices and how are they related to a firm's network dependencies?

As Brealey and Myers (2000) have pointed out, the value of a firm coalition does not necessarily exceed the sum of the firm values that form it and hence the classical coalition formation results, such as the Shapley (1953) value or the results in Maskin (2003), which all assume the superadditivity of the value function, are not appropriate in this environment. Superadditivity is a very restrictive assumption implying that the value of *any* coalition is necessarily greater than the value of its parts. Games with externalities, mergers for diversification and many others are all examples where the superadditivity axiom fails. Moreover, there exists considerable empirical literature on the tradeoff between the benefits of mergers and over-diversification. Ravenscroft and Scherer (1988) argue that the productivity declines in the years following the merger, while others, such as Healy, Palepu, and Ruback (1992) found an average increase in subsequent corporate returns.

To account for this deficiency of existing theories we develop a theory of coalition formation without the superadditivity axiom. The theory that we propose functions as follows. First, the firms enter the bargaining process sequentially, i.e. firm i starts bargaining when firms $1, \ldots, i-1$ have already formed the coalitions. The coalitions can not be re-negotiated and can not break apart - our model does not apply to spinoffs. Firm i joins the coalition which maximizes the total welfare and is paid its marginal contribution to the share price increase (Vickrey principle). The process is repeated for all possible permutations, i.e. for all bargaining orders. The axioms of the theory can be implemented algorithmically but are deeply recursive and resource consuming. We prove that by adding the value function superadditivity to the above stated axioms we obtain the Shapley value again. Therefore our proposal can be considered as an extension of the Shapley concept to coalitional games without superadditivity.

Since the analysis of agency conflicts *within* the firm is not the main purpose of this paper, we assume that the bargaining parties are managers on behalf of shareholders and that their incentives are aligned. After (and if) the merger occurs, one of the managers of the merging firms is paid out and leaves all the managerial decisions to the other one. The merged firm managerial structure is then the same as before the merger. Since no distinction is made between the firm and its shareholders (managers), we say with the slight abuse of language that the bargaining agents are firms themselves.

We then apply the newly developed bargaining theory to a model of firm merger in a network of firms, where economic dependencies between firms are static[1]. Firms issue buy orders, as described by network dependencies, that generate the asset dynamics as in Brumen and Vanini (2006). We assume that the equity value maximization is the firm's/coalition's sole objective function and in order to capture the tradeoff between the cash flow volatility and the possibility of default we use the Leland and Toft (1996) model of equity pricing. By relying on the buyer-supplier network theory, the model has the potential to incorporate all three groups of mergers: vertical, horizontal and conglomerate, and give predictions as to which merger is more likely in a particular network type.

[1] An equivalent statement is that the cost of breaking an economic relationship is too high, see Grossman and Hart (1986).

The results show that firm values in a network should not be considered in isolation. “Merger correction” is the difference between the firm’s equity value with and without accounting for merger possibilities. This fact is documented in Elgers and Clark (1980) who observe that moderate gains to buyer firms and substantial gains to seller firms occured in the period of approximately three months prior to the merger. Higher network dependence decreases the number of coalitions formed, i.e. it generates merger activity, higher firm equity values and increasing merger surplus. Our model therefore predicts that firms suffering high and negatively correlated levels of cash flow volatility should merge and exactly the opposite should happen for firms with positively correlated cash flows. The average leverage ratio in the economy decreases the number of coalitions in a network but only up to a certain point after which the merger activity decreases again, thereby generating a non-symmetric inverted U-shaped merger activity curve with more mergers occuring for high than for low leverage ratios. The results suggest that in networks where the number of buyers (resp. suppliers) dominates the number of suppliers (resp. buyers), mergers occur primarily in the group that dominates the other in the number of members.

The neoclassical merger theory of Jovanovic and Rousseau (2002) relates the merger and acquisition activity to firm’s (Tobin’s) Q value and technological changes. In our paper we show that the mergers occurs even if there is no change in technological efficiency of individual firms and the merger serves as a hedge against volatile cash flows. More recently, the papers by Lambrecht (2004), Morellec and Zhdanov (2005), Lambrecht and Myers (2006) and Thijssen (2008) proposed a real options merger formation framework based on the comparison between the underlying real activity of the individual and the merged firm. The feature of all of these models is that eventually only a single firm would exist[2]. The models stated above all have the following shortcomings. Firstly, they are limited to two firms and hence the results oscillate wildly between no mergers and the merger of all (two) firms in the economy. An economic environment of multiple firms enlarges the merger possibility and the coalition formation opportunity set. Secondly, the papers by Lambrecht, Morellec and Myers are of infinite horizon. While mergers can produce long-term benefits, they can end up in default over the short run. Finally, the existing literature can not explain merger waves. Contrary to the articles by Lambrecht (2004) and Morellec and Zhdanov (2005), our model predicts multiple coalitions (as opposed to a single one) and cyclical merger activity in line with Lambrecht (2004). Dynamic programming applied to our model has the potential to extend the merger decisions into the time domain, similar to Thijssen (2008), and even to the analysis of mergers/spinoffs. This is presented in the final chapters of the paper. The papers by Scharfstein and Stein (2000), Inderst and Laux (2006) and others take a more strategic approach to the formation of internal capital markets on behalf of division managers and the incentives that form them. A distantly related paper is Habib and Mella-Barral (2006) which investigates different type of firm-firm connections, such as partnerships and mergers and similarly to our paper identifies conditions for a specific form of firm relationship. Our paper also provides an explanation for managers’ decision about the “currently unprofitable mergers”, as in Gorton and Rosen (2005), which can generate profits in the long run. The paper shares insights with Shleifer and Vishny (2003) where stand-alone companies can increase their stock price by merging or acquiring other firms.

This paper is structured as follows. Section 2.2 illustrates the merger theory in a simple two firm buyer-supplier example. Section 2.3 develops the general theory of coalitional games without the

[2] In their models, the merger is motivated by the increase of a specific firm characteristic, such as a sales price, driven by the geometric Brownian motion, to a high enough level. This is a property of the geometric Brownian motion process. Other processes governing the real economic activity would most probably generate different merger predictions.

superadditivity axiom which is then applied as a merger solution concept. In Section 2.4 the merger theory in a general network environment is presented. Section 2.5 sums up the results.

2.2 The two firm case

We illustrate the merger behavior and firm stock price formation on a simple two firm network with dependent firms. We assume the setting of Brumen and Vanini (2006) of a buyer (firm 1) and a supplier (firm 2), see Figure 2.1(a). The buyer firm issues buy orders modelled by the jumps of a Poisson process N_1 (with intensity λ_1). A jump of N_1 induces a *net* cash flow to the supplier (firm 2) in the amount $E_{12}\,P_2\,A_2$ where E_{12} is the number of links between firms 1 and 2 and denotes the strength of business relationship between the firms. P_2 is the proportionality factor describing the net cash flow proportion of the total assets (A_2) paid by the buyer to the supplier firm. In addition to network buyer-supplier cash flows, the two firms have external cash flows, irrespective of the network. The dependence[3] between the external cash flows and the buy/supply orders is represented in a diagonal matrix $\boldsymbol{B} = \text{diag}\,(B_1, B_2)$. External cash flows are depicted dashed in Figure 2.1(a). The element B_1 describes the dependence between the buy orders and external cash flows of firm 1. For every buy order the firm 1 receives an external cash flow in the amount $B_1\,P_1\,A_1$. The external cash flow received by firm 1 is proportional to its size A_1 where the proportionality factor is P_1. The number B_1 describes how many times this amount firm 1 receives from external sources. Additional Poisson process N_2 (with intensity λ_2) drives the external cash flows of firm 2. The distinction between external and network cash flows allows us to focus and study specific firm-firm relationships in greater detail and model other economic influences on the firms statistically. The dynamics of the firms' asset values can be written as (see Brumen and Vanini (2006), Proposition 2.1)

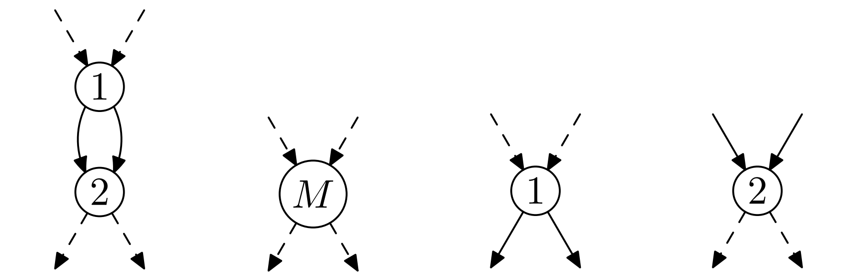
$$dA_1 = A_1(\eta_1\,\mathrm{d}t + B_1 P_1\,\mathrm{d}N_1) \tag{2.1}$$
$$dA_2 = A_2(\eta_2\,\mathrm{d}t + E_{12} P_2\,\mathrm{d}N_1 + B_2 P_2\,\mathrm{d}N_2) \tag{2.2}$$

Some empirical support for the high-frequency version of this specification (see Brumen and Vanini (2006), Proposition 2.5) is given in Hackbarth and Morellec (2008). We assume that both firms have issued equity and zero-coupon debt with principals D_1, D_2 and maturity T. We denote by S_1, S_2 the equity values. In a two firm network we define the *dependency value* as the number of economic links between the firms, i.e. the value of E_{12}. We use Leland and Toft (1996) to determine the value of stocks. In this setting the firms' equity values can be expressed as follows.

Proposition 2.2.1. *The equity price of firm i with principal value D_i and maturity T is given by $S_i = S_i(\sigma_i)$, where $\sigma_1 = (B_1^2 P_1^2 \lambda_1)^{1/2}$, $\sigma_2 = (E_{12}^2 P_2^2 \lambda_1 + B_2^2 P_2^2 \lambda_2)^{1/2}$, and S_i is the Leland/Toft equity value as given in Leland and Toft (1996), equation (9).*

The choice of the Leland and Toft (1996) model correctly reflects the tradeoff between the increased stock price due to increased volatility of cash flows and the negative effect due to increased likelihood of default. The results of Leland and Toft (1996) are restated in the Appendix, Proposition 2.A.4.

[3]Since the buy-supply orders are driven by the Poisson processes the dependence between the external and network cash flows is not Gaussian and the term correlation is not an appropriate one. In the approximated Gaussian case one can consider $\boldsymbol{B}$ as the correlation matrix between external and network cash flows.



(a) Original network.

(b) Cash flows in a merged network.

(c) Cash flows from the perspective of firm 1.

(d) Cash flows from the perspective of firm 2.

Figure 2.1: Example of firms and net cash flows in a two firm buyer - supplier network - 2.1(a). The network cash flow directions are denoted in solid and the external cash flows in dashed lines. A merged firm 2.1(b) possesses only external cash flows. Cash flows in a network 2.1(a) as seen by individual firms 1 and 2 are presented respectively in figures 2.1(c) and 2.1(d).

Now consider the case when the two firms merge. We assume additionally that there are no agency conflicts and the shareholder value maximization is the only motive of the firm. The merged firm operates the individual units as independent entities, with the difference that the transfer of goods between the firms is now an internal capital market. The dynamics of the merged company is then given as $A_{12} = A_1 + A_2$, where A_1 and A_2 are defined as in (2.1)-(2.2) and the equity value of the merged company is $S_{12} = S_{12}(\sigma_{12})$, where $A_{12}(0) = A_1(0) + A_2(0)$, $D_{12} = D_1 + D_2$, $\sigma_{12} = \sqrt{(K_1(B_1P_1 + E_{12}P_2))^2\lambda_1 + (K_2B_2P_2)^2\lambda_2}$, $K_1 = 1 - K_2 = \frac{A_1(0)}{A_1(0)+A_2(0)}$ and S_{12} is the same as in Proposition 2.2.1 the Leland and Toft equity value. The factor K_1 (K_2) represents the intial relative asset size of firm 1 (2) to the total assets of all firms in the economy. We notice that the single most important parameter of the merged firm that drives the stock values is the volatility difference between the individual firms and the merged one.

Proposition 2.2.2. *In a two firm example the firms merge if and only if $S_{12} \geq S_1 + S_2$. In this case, share prices of firms 1 and 2 are $\frac{1}{2}(S_1 + S_{12} - S_2)$ and $\frac{1}{2}(S_2 + S_{12} - S_1)$ respectively. If $S_1 + S_2 \geq S_{12}$ then firms 1 and 2 do not merge and are worth S_1 and S_2 respectively.*

Figure 2.2 shows the equity value S_{12} of the merged firm (dotted line) and the sum of equity values of both firms (solid line) for different values of network dependency parameter E_{12}. The merger is rejected at low levels of network dependency, the “no merger part” in Figure 2.2, where the sum of individual stock values $S_1 + S_2$ is higher than the stock of the merged firm S_{12}. For this range of E_{12} the negative external cash flows B_2 of the buyer (firm 2) would decrease the stock price of the merged firm and the merger is rejected by the buyer firm 1. For high levels of network dependency parameter E_{12} (the “merger part” in Figure 2.2) the default effect of volatile cash flows starts to dominate in both firms and it is here where the firms merge and form an internal capital market in the buyer-supplier chain. In this range of network dependency values, the positive effect of volatility decrease through a merger dominates and the merger is preferred for both firms. Figure 2.2 also shows that increased network dependence in the buyer-supplier chain raises the value of the merger, i.e. the difference between the merged firm value S_{12} and the sum of individual firms $S_1 + S_2$ increases with E_{12}. Different values of network/leverage parameters move and scale the two functions in Figure 2.2 and it is possible that merger is rejected for high dependency values as well as approved for low ones.

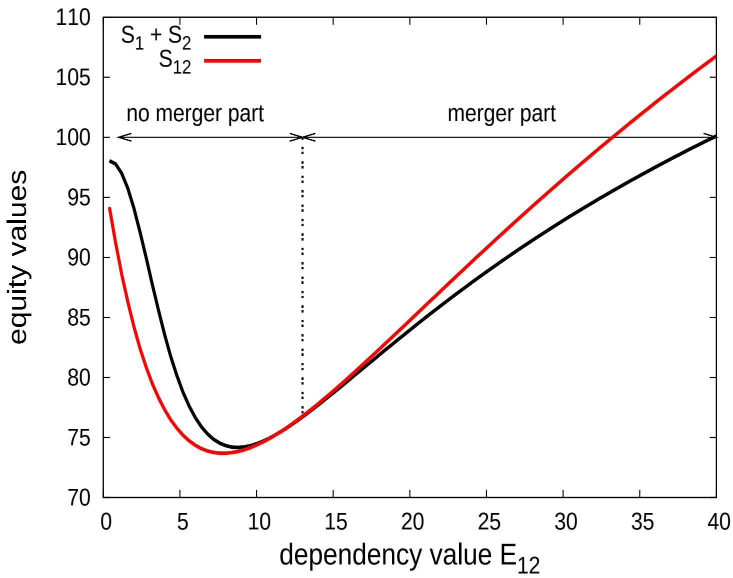


Figure 2.2: A representation of a merger indicator with respect to the dependency value of the network. The dependency value of the network was defined in Brumen and Vanini (2006), Section 4, and in the case of two firms reduces to the number of connections E_{12} between firm one and two. Proposition 2.2.2 indicates that the firms merge when $S_1 + S_2 \leq S_{12}$. The model parameters are as follows: the buy order proportionality factor $P = 0.0050$, the buy order intensities are $\underline{\lambda} = (70, 80)'$, external cash flow matrix $\boldsymbol{B} = \text{diag}(9, -3)'$, recovery rate 0.7, $\underline{A}_0 = (100, 120)'$ and the principal debt values are $\underline{D}_0 = (40, 40)'$.

We next examine the behavior of the network dependency cut-off values, i.e. the values of the network dependency parameter E_{12} when the merger becomes profitable, on the firms' average leverage ratio L computed for a two firm example as

$$L = \frac{1}{2}\left(\frac{D_1}{S_1} + \frac{D_2}{S_2}\right), \tag{2.3}$$

where D_1, D_2 are the principal debt values (both firms have issued a zero-coupon bond and equity), $A_1(0), A_2(0)$ the initial asset values and $S_1 = A_1(0) - D_1$, $S_2 = A_2(0) - D_2$ the equity values of firms 1 and 2 respectively. The extension to multiple firms is obvious. Figure 2.3 shows that the network dependency cutoff value E_{12}^* decreases as the average leverage ratio L increases but only up to a certain point (around 0.7 in Figure 2.3). For L in the interval $[0.25, 0.7]$ the firms merge sooner, that is at lower levels of network dependency E_{12}. For L above 0.7 the merger activity decreases again. The shape of the curve can be economically interpreted as follows. Increasing the average L ratio increases the probability of default for every individual firm in the network. A merger therefore becomes more attractive. After L grows above certain high enough level, the probability of default is already so high that only additional increase in cash flow volatility can save the firms from defaulting. This parallels the asset substitution effect. The network dependency cutoff value E_{12}^* displays an U-shaped curve and the merger activity an inverted U-shaped line. The maximum number of mergers does not occur for extremely low or high leverage ratios, even though in both states mergers do occur. We emphasize that the "middle" leverage ratios in the Figure can be obtained by a combination of high leverage of one firm and a low leverage of the other firm, from which we conclude that the merger is more likely between firms of heterogenous leverage ratios.

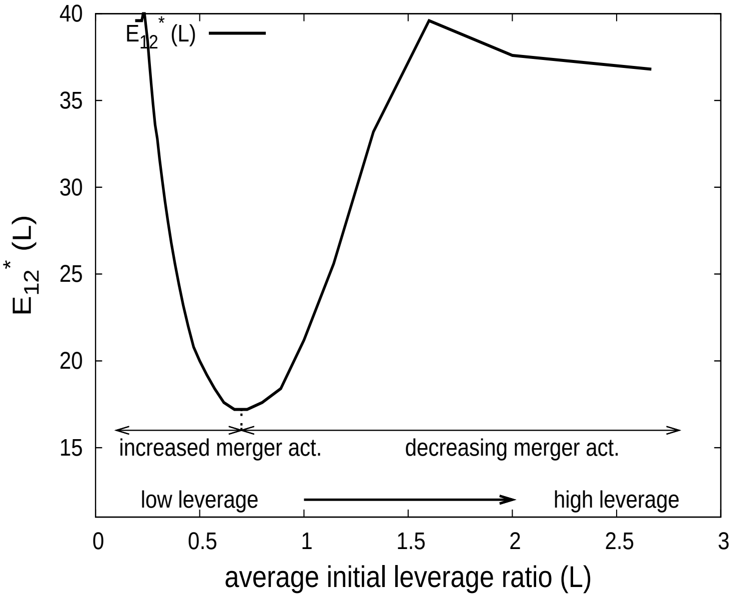


Figure 2.3: The relationship between the network dependency cutoff values E_{12}^{*}, i.e. the value of the network dependence parameter E_{12} where merger becomes rationable, and the average initial leverage ratio L. Other parameters of the model are given in the caption below Figure 2.2.

The existence of mergers for very high leverage ratios has been empirically documented in Bernile, Lyandres, and Zhdanov (2007).

2.3 Theory of Coalition Formation without Externalities and the Superadditivity Axiom

In Section 2.2 we have analyzed the merger process between two firms. The decision in this case was simple - if the value of the merged firm's stock is higher than the combined stock value of individual firms the incentive to merge exists. The decisions are not so simple already in the case of three firms. Firm 3 has the possibility of forming a coalition with firm 1 or firm 2 or, if the firms 1 and 2 merged, with the merged firm. Firms 1 and 2 optimally anticipate the behavior of firm 3 and act accordingly. Furthermore it is entirely unclear how the increased share values of the merger should be split among the merging parties. Another problem is that the worth of a coalition (such as a merger/takeover/partnership) of two firms may be worth less than the sum of individual entities. Brealey and Myers (2000) give an example of Kaiser Industries, a holding company for Kaiser Steel, Kaiser Aluminum and Kaiser Cement which traded at significant discount until it sold off its holdings again forming separate companies. Therefore the classical theory of coalition formation first proposed by Shapley (1953) does not capture the issue well. In order to resolve these issues we need some game theoretic concepts which provide consistent guidance into the firm's merger decisions in a multi-firm economies.

In this section we propose a new game theoretic solution concept which addresses the problems of the Shapley theory, the empirical merger puzzles and is algorithmically easily implementable. The proposed theory establishes a solution concept for cooperative games without externalities and without the superadditivity axiom[4]. Since the superadditivity axiom does not hold the grand coalition does not necessarily form. If in addition the value function satisfies also the superadditivity assumption, the solution concept coincides with the Shapley value. We therefore consider the proposed solution concept as a (particular) extension of the Shapley value to games with general value functions.

We denote by $\overline{n} = \{1, \ldots, n\}$ the set of integers smaller than $n+1$. An *N-player transferable utility game (N, v) without externalities* is given by

- A set of players $\overline{N}$.
- A function v, which assigns a worth $v(S)$ to a coalition $S \in \mathcal{P}$, given a partition $\mathcal{P}$ of $\overline{N}$.

We fix the number of players N and focus on coalitional games that are normalized to $v(\emptyset) = 0$. We call a partition $\mathcal{P}^n$ of $\overline{n}$, $n < N$ a *partial partition*. Given a partial partition $\mathcal{P}^{i-1}$, let $\varphi_i(\mathcal{P}^{i-1})$ be the prediction function of player's i payoff and $\psi(\mathcal{P}^{i-1})$ be the prediction of a partition of $\overline{N}$ given that the partition of $\overline{i-1}$ is $\mathcal{P}^{i-1}$. We require that $\{\varphi_i\}_{i=1}^N$ and ψ satisfy the following set of axioms. The axioms are inspired heavily by Maskin (2003), but also differ from it substantially.

(NA) *Non-negotiation commitment:* Let $j, k \in \overline{i}$, $i < N$, and partial partition $\mathcal{P}^i$ be given where $j \in S \in \mathcal{P}^i$ and $k \notin S$. If $j \in S^* \in \psi(\mathcal{P}^i)$ then also $k \notin S^*$.

The axiom NA guarantees that the players that have been allocated to separate coalitions in the bargaining process do not "renegotiate" with other players entering the bargaining process after him to form a new coalition or join a different one than the already assigned. Maskin interprets this axiom as "cutting the telephone lines".

(BC) *Binding coalitions:* Let $i < N$ and the partial partition $\mathcal{P}^i$ be given. If $S \in \mathcal{P}^i$ then there exists $S' \in \psi(\mathcal{P}^i)$ such that $S \subset S'$.

The axiom BC assures that as the bargaining process proceeds the already formed coalitions do not break apart, i.e. players in a coalition remain together in that coalition till the end of the bargaining process.
For any $S \in \mathcal{P}^{i-1} \cup \emptyset$ and $\hat{S} \in \mathcal{P}^{i-1} \cup \emptyset'$ we define

$$\Phi^i(S, \hat{S} | \mathcal{P}^{i-1}) = \begin{cases} v(S^N(S)) - \sum_{j>i} t_j(S^{j-1}(S) | \psi^{j-1}(\mathcal{P}^{i-1}, \hat{S} \cup i)) & S \neq \hat{S} \\ v(S^N(S \cup i)) - \sum_{j>i} t_j(S^{j-1}(S \cup i) | \psi^{j-1}(\mathcal{P}^{i-1}, S \cup i)) & S = \hat{S} \end{cases} \tag{2.4}$$

where $t_j(S|\mathcal{P})$ is defined recursively in the VP axiom below (equation (2.8)). We used the following notation: $\psi^j(\mathcal{P}^i)$ $(j \geq i)$ is the partition of $\overline{j}$ such that the elements of $\psi^j(\mathcal{P}^i)$ are members of $\psi(\mathcal{P}^i)$ without players $\overline{N} \backslash \overline{j}$. $S^j(S)$ is the unique element of $\psi^j(\mathcal{P}^i)$ such that $S \subset S^j$. The existence of $S^j(S)$ is guaranteed by the BC axiom. We note that

$$\Phi^N(S, \hat{S} | \mathcal{P}^{N-1}) = \begin{cases} v(S) & S \neq \hat{S} \\ v(S \cup N) & S = \hat{S} \end{cases} \tag{2.5}$$

[4] Under the superadditivity of the value function we mean that $v(S_1) + v(S_2) \leq v(S_1 \cup S_2)$ for any coalitions $S_1 \cap S_2 = \emptyset$.

The quantity $\Phi^i(S, S'|\mathcal{P}^{i-1})$ is the worth of the coalition $S \in \mathcal{P}^{i-1}$ if the first $i-1$ players form a partial partition $\mathcal{P}^{i-1}$, i.e. they have already formed selected coalitions, and i is allocated to S'. Φ^i captures the optimal anticipation by player i of future players $j \geq i$ behaving optimally themselves. Since the game does not have externalities the function Φ^i does not depend on $\mathcal{P}^i$. We prove this formally in the Appendix, Lemma 2.A.1. We maintain the dependence on $\mathcal{P}^i$ in oder to keep track of the coalitions formed.

(CA) *Competitive allocation:* Let $\mathcal{P}^{i-1}$ be a partial partition. Then i is allocated to $S^o \in \mathcal{P}^{i-1}$, such that

$$S^o = \underset{\hat{S} \in \mathcal{P}^{i-1}}{\operatorname{argmax}} \sum_{S \in \mathcal{P}^{i-1}} \Phi^i(S, \hat{S}|\mathcal{P}^{i-1}), \tag{2.6}$$

i.e. $(S^o \cup i) \in \psi^i(\mathcal{P}^{i-1})$ if and only if S^o is the maximizer in (2.6).

CA axiom states that the player i joins the coalition which generates maximum total welfare of all coalitions. This is the “efficiency axiom”.

(CW) *Competitive wages:* Let $\mathcal{P}^{i-1}$ be a partial partition and assume that player i is competitively allocated (i.e. according to axiom CA) to coalition $S^o \in \mathcal{P}^{i-1}$. Then the payoff to player i is

$$\varphi_i(\mathcal{P}^{i-1}) = \Phi^i(S^o, S^o|\mathcal{P}^{i-1}) - \Phi^i(S^o, S^{oo}|\mathcal{P}^{i-1}), \tag{2.7}$$

where $S^{oo} \in \mathcal{P}^{i-1}$ is any coalition $S^{oo} \neq S^o$. The choice of S^{oo} is arbitrary due to the fact that the game does not have externalities.

The CW axiom speaks about the value assigned to a player in the bargaining process. The player receives the difference between the coalition's worth with that player $\Phi^i(S^o, S^o|\mathcal{P}^{i-1})$ and the one when the player joins any other coalition S^{oo} - $\Phi^i(S^o, S^{oo}|\mathcal{P}^{i-1})$. The player is awarded his marginal contribution to coalition's worth. In our setting, that is in games without externalities, S^{oo} is well defined. The choice of S^{oo} is a major obstacle in defining the solution concept for cooperative games with externalities.

(VP) *Vickrey payments:* Let $\mathcal{P}^{i-1}$ be a partial partition and player i be competitively allocated to $S^o \in \mathcal{P}^{i-1}$. Then for all $S \in \mathcal{P}^{i-1}$ the following holds

$$t_i(S|\mathcal{P}^{i-1}) = \begin{cases} \Phi^i(S^o, S^o|\mathcal{P}^{i-1}) - \Phi^i(S^o, S^{oo}|\mathcal{P}^{i-1}) & S = S^o \\ 0 & S \neq S^o \end{cases}, \tag{2.8}$$

where S^{oo} is as in the CW axiom.

The last axiom states that only the coalition which the player joins awards that player with the exact amount by which the coalition's value increases. This is in stark contrast to games with externalities, where externalities can induce payments from other coalitions, and greatly simplifies the analysis.

The bargaining process proceeds as follows. Players enter the bargaining process sequentially, i.e. player $i \leq N$ bargains after players $1, \ldots, i-1$ have already formed their respective coalitions. Player i bargains for its payoff (axiom CW) and joins the selected coalition (CA). The coalition awards the player its marginal contribution according to the VP axiom. The stability of the coalition formation is guaranteed by the NA and BC axioms. The axioms NA-VP are constructive and can be implemented on a computer. The algorithm that implements them is unfortunately highly recursive and resources consuming. The bargaining and coalition formation process arising from the axioms above bears close resemblance to the Shapley value. In the case of more than one coalition outstanding, the player next in the bargaining process evaluates his contribution to every coalition (including forming a new one) correctly anticipating the decisions of other players entering the bargaining process later. The bargaining with each coalition separately is done in exactly the same manner as in Shapley. We illustrate this on an example of 5 players with the already formed coalitions being $\{\{1,2\},\{3\}\}$. In this scenario player 4 enters the bargaining process next. The decision made by player 4 is whether to join the first or the second coalition or to start his own. In both of the three cases he bargains in a manner described by Shapley correctly anticipating whether player 5 joins the same coalition or any other. Player 4 then joins the coalition to which his marginal contribution is greatest.

The following theorem provides the existence of a payoff prediction function φ and a coalitional prediction function ψ that satisfy above axioms.

Theorem 2.3.1. *For every N-player transferable utility game (N,v) there exist unique payoff prediction function φ and coalition prediction function ψ as defined above, which satisfy axioms NA, BC, CA, CW and VP.*

A feature of the proposed solution concept above is that the players entering the bargaining process can join inefficient coalitions temporarily for the benefit of more efficient outcomes in a coalition with other players later, i.e. bargaining order has an effect on coalition formation. Example of such a three player game is given in the Appendix, Proposition 2.A.3 where the solution concept in Theorem 2.3.1 predicts a grand coalition of three players, yet a a restriction to the first two players results in separate coalitions for both agents. When forming coalitions the proposed solution concept considers all players in the bargaining process even though the bargaining takes place sequentially.
To construct the payoff prediction function we randomize over the order of players entering the bargaining process. Let $\pi\in\mathcal{S}^N$ be a permutation in a set of all permutations of N elements. We define

$$\varphi_i=\frac{1}{N!}\sum_{\pi\in\mathcal{S}^N}\varphi_i^\pi, \tag{2.9}$$

where φ_i^π is obtained as before when the order of players entering the bargaining process is π. The predictor of the coalition structure is a uniform distribution over all the coalition predictions, i.e. $\psi_i=\psi_i^\pi$ ($\pi\in\mathcal{S}^N$) with probability $\frac{1}{N!}$.

The justification for the selection of axioms above comes additionally from the following theorem.

Theorem 2.3.2. *Let v be a coalitional game of N players. The following holds.*

(a) For every $S\in\psi(\{\emptyset\})$ the efficiency equation $\sum_{i\in S}\varphi_i(\psi^{i-1}(\{\emptyset\}))=v(S)$ holds.

(b) For every partial partition $\mathcal{P}^{i-1}$ we have that $\sum_{S\in\mathcal{P}^{i-1}}t_i(S|\mathcal{P}^{i-1},S^o)=\varphi_i(\mathcal{P}^{i-1})$, where $S^o\in\psi^i(\mathcal{P}^{i-1})$.

(c) If for some i and for every subset $S\subset\{1,\dots,N\}$ of N players $v(S\cup\{i\})=v(S)$ holds, then $\varphi_i(\mathcal{P}^{i-1})=0$.

(d) *If in addition to axioms NA, BC, CA, CW and VP, the superadditivity of the value function v is also assumed, then φ_i in equation (2.9) equals the Shapley value of player i and the grand coalition forms.*

Part (a) of the theorem states that the sum of payoffs to all coalition members precisely equals the worth of that coalition, i.e. the worth of the coalition is divided among the players that form it and there are no loses due to the bargaining process. It is the marginal contribution of every player (the CW axiom) that determines the percentage of the coalition's worth the player receives. Part (b) states that the payoff to player i is the sum of all Vickrey payments by all existing coalitions. It is obvious from the definition of the Vickrey payments that only the coalition that the player joins pays that player. Intuitively, this is a consequence of the no externality assumption of the game. Part (c) is an extension of the Shapley axiom to coalitional games without superadditivity. It states that the player who has zero marginal contribution to all coalitions receives zero payoff. Part (d) proves that the established theory is a generalization of the Shapley value concept.

2.4 A general model

We now turn to the case of N firms in a general buyer-supplier network. We use the same network structure and notation as in Brumen and Vanini (2006). A network consists of N firms who issue buy orders to their suppliers. Buy orders depend on business relationships between firms described by the adjacency matrix $\tilde{\boldsymbol{E}} \in \mathbb{R}^{N\times N}$. For example, $\tilde{E}_{12} = 2\tilde{E}_{23}$ means that the firm 1 supplies at every issue of a buy order twice as much (proportional to its asset value) to firm 2 as does firm 2 to firm 3. Buy orders of firm i arrive with intensity λ_i independently of all other firms ($\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^{N\times N}$). Every buy order induces a *net* cash flow to the suppliers in the amount proportional to its outstanding assets where the proportionality factor equals P_i, $\boldsymbol{P} = \mathrm{diag}(P_1, \dots, P_N) \in \mathbb{R}^{N\times N}$. Empirical support for this kind of model is given in Brumen and Vanini (2006). The supplier i to the buyer j is characterized by the condition $\tilde{E}_{ij} > 0$, $j = 1, \dots, N$. In addition to network generated cash flows, firms receive external cash flows where the correlation between the network generated cash flows and the external ones is given by the matrix $\boldsymbol{B} \in \mathbb{R}^{N\times N}$. We denote by $\boldsymbol{E} = \boldsymbol{B} + \tilde{\boldsymbol{E}}$. Under certain technical assumptions (see Brumen and Vanini (2006), Theorem 3.1 for precise statements) the asset value process $\underline{A} = (A_1, \dots, A_N) \in \mathbb{R}^N$ of all firms in the network can under the risk-neutral measure be approximated by a multivariate geometric Brownian motion

$$\mathrm{d}\underline{A} = \boldsymbol{A}\left(r\underline{1}\,\mathrm{d}t + \boldsymbol{P}\boldsymbol{E}\boldsymbol{\lambda}^{1/2}\,\mathrm{d}\underline{W}\right), \tag{2.10}$$

where $\underline{1}$ is a vector of ones and $\underline{W}$ is a multivariate Brownian motion both of dimension N. This is the same assumption as in Hackbarth and Morellec (2008). The asset volatility of firm $1 \le i \le N$ in this setting is given by

$$\sigma_i^2 = \sum_{j=1}^{N} E_{ij}^2 P_i^2 \lambda_j \tag{2.11}$$

that is it depends on all of its buy orders E_{ij} weighted by the proportionality factors P_i and buy order intensities λ_j.

We define *firm coalition* as the cooperation of firms in forming internal capital markets. Classical mergers, mergers of equals, acquisitions or hostile takeovers are in our setting all examples of coalition formation. We make the following assumptions. The objective function of a firm coalition is its equity value maximization. There are no agency conflicts within the coalitions and no tax advantages of a bigger coalition. Finally, firms in a coalition operate as subdivisions of that coalition, i.e. there are no production efficiency gains and no externalities of firm mergers. A merger changes the volatility of cash flows (and therefore the equity prices) through the creation of internal capital markets, but does not effect any other coalition. We use Leland and Toft (1996) equity pricing model to determine the equity value of coalitions.

The bargaining and coalition formation process described in Section 2.3 is done at the beginning of the period. After the coalition formation, no subsequent mergers or coalition breakups are allowed[5]. We first specify the value of coalitions. Let $LT(A,P,\sigma)$ be the Leland-Toft equity value of the firm with beginning of the period firm market value A, total principal P and asset volatility σ as defined in Proposition 2.A.4 of the Appendix. Assume $S \subset \overline{N}$ is a coalition of firms. The value $v(S)$ of this coalition is given by

$$v(S) = LT\left(\sum_{i\in S} A_i(0), \sum_{i\in S} D_i, \sqrt{\sum_{j\in S}\left(\sum_{i\in S} K_i E_{i,j} P_i\right)^2 \lambda_j}\right),$$

where $K_i = \frac{A_i(0)}{\sum_{l\in S} A_l(0)}$, i.e. the initial coalition asset market value is the sum of all assets in the coalition $\sum_{i\in S} A_i(0)$ and the debt principal of the coalition is the sum of debt principals of individual firms $\sum_{i\in S} D_i$. The volatility of the coalition is the weighted average of individual firm volatilities weighted by their relative initial sizes $\frac{A_i(0)}{\sum_{l\in S} A_l(0)}$. We interpret the value $v(i) = LT(A_i(0), D_i, \sigma_i)$ for any $1 \le i \le N$ as the the stand-alone firm value with σ_i as in (2.11). The axioms NA, BC, CA and VP of the previous section all retain their meaning in coalitions. The bargaining process proceeds as follows. Firms enter the bargaining process sequentially, i.e. firm $i \le N$ begins negotiation after firms $1,\ldots,i-1$ have already formed the appropriate coalitions. Firm i's value is determined by the CW axiom, taking into account future coalition building, and joins the selected coalition according to the CA axiom. The stability of the coalition formation is guaranteed by the NA and BC axioms.

As an illustration we consider a network of 3 firms. A larger network complicates computations considerably but does not offer any new economic insights. In this network firm 1 is a supplier, i.e. a seller to both firms 2 and 3, firm 2 is an intermediary (a buyer from firm 1 and a supplier of the retailer) and firm 3 is a retailer (buys from both other firms). The parameter values described before are given in Table 2.1. The parameters in Table 2.1 are chosen to reflect realistic values of such a

$\boldsymbol{E}$	diag($\boldsymbol{P}$)	$\underline{\lambda}$	diag($\boldsymbol{B}$)	$\underline{D}$	$\underline{A}_0$	T
$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0050 \\ 0.0070 \\ 0.0030 \end{bmatrix}$	$\begin{bmatrix} 100 \\ 110 \\ 115 \end{bmatrix}$	$\begin{bmatrix} -5.000 \\ -0.500 \\ 5.000 \end{bmatrix}$	$\begin{bmatrix} 70 \\ 80 \\ 90 \end{bmatrix}$	$\begin{bmatrix} 100 \\ 110 \\ 120 \end{bmatrix}$	1

Table 2.1: Model parameters for the network of three firms. $\boldsymbol{E}$ is the network dependency matrix, $\boldsymbol{P}$ the proportionality matrix and $\boldsymbol{B}$ the correlation between network and external cash flows, as explained in the beginning of Section 2.4. $\underline{\lambda}$ is the buy order intensities vector of firms in the network. $\underline{D}$ is the principal of the zero-coupon bonds issued by firms in the network. Vector $\underline{A}_0$ shows the firms' market asset values at time 0. The time period ends at T.

[5]This assumption can be relaxed and we comment on it in Section 2.4.2.

small economy. The D/E ratios of firms 1, 2 and 3 are between 2 and 3 and the asset volatilities are between 0.2 and 0.4 for all three firms.

We answer the following two questions. What is the effect of network dependency on the merger activity and economic surplus generated by mergers? Secondly, what effect does firm leverage have on merger activity?

The effect of different levels of network dependencies on coalition formation is shown in Table 2.2, where in column 9 we present only the average number of coalitions formed for all bargaining orders (there are a total of $3! = 6$ orders for this case). The average number of 3 coalitions means that no mergers have occurred. All firms operate separately. In cases when 2 coalitions have formed, two of the firms have merged and the third one operates independently. We deduce which firms have merged by looking at their values with and without the merger effects, columns $2-4$ and $5-7$ respectively. The difference between a firm's value in isolation (without merger effects) and its merged value proxies for the Leland and Toft (1996) volatility/default cash flow tradeoff on the account the creation of internal capital markets and is presented in column 8. The increase in firm value is attributed to the positive effect between the tradeoff of firm default probability and stock increase due to upward potential of volatile cash flows. Looking at the firm values under merger

Network dependency	Firm values with mergers			Stand-alone firm values			Merger surplus	Avg. # of coalitions	Coalitions formed
M	1	2	3	1	2	3			
1.00	12.82	61.18	246.35	12.82	61.18	246.35	0.00	3.00	(1),(2),(3)
2.00	6.07	5.42	246.35	1.66	1.01	246.35	8.82	2.00	(1,2),(3)
3.00	16.48	19.16	246.35	0.29	2.97	246.35	32.38	2.00	(1,2),(3)
4.00	25.02	32.35	246.35	3.51	10.84	246.35	43.01	2.00	(1,2),(3)
5.00	32.43	43.07	246.35	8.23	18.87	246.35	48.39	2.00	(1,2),(3)
6.00	38.89	51.78	246.35	13.25	26.14	246.35	51.27	2.00	(1,2),(3)
7.00	44.53	58.98	246.35	18.12	32.57	246.35	52.82	2.00	(1,2),(3)
8.00	49.47	65.05	246.35	22.68	38.25	246.35	53.59	2.00	(1,2),(3)
9.00	53.82	70.21	246.35	26.90	43.29	246.35	53.84	2.00	(1,2),(3)
10.00	57.64	74.65	246.35	30.79	47.79	246.35	53.71	2.00	(1,2),(3)

Table 2.2: The dependence of firm stock values and the number of coalitions formed with respect to the dependency value of the network. The network dependency of the whole network increases with ascending M, i.e. a network dependency matrix in row M is $\boldsymbol{E}(M) = M\boldsymbol{E}$, where $\boldsymbol{E}$ is given in Table 2.1. The next three columns (2-4) present the value of firms 1-3 incorporating the synergy effect of the mergers. The next three columns (5-7) are the values of individual firms, not incorporating the merger effects. Column 8 gives the average number of coalitions formed and the last column shows the coalitions most likely to form keeping in mind that for different firm orderings in the bargaining process different coalitions can arise. Other parameters are given in Table 2.1.

possibilities we notice that: a) Merger surplus increases with network dependency parameter M but approaches an upper bound, see column 8, Table 2.2. b) Higher levels of network dependency induce the supplier and the intermediary (firms 1 and 2) to merge but not the retailer (firm 3), column 10 in Table 2.2. c) The retailer is unaffected by the network dependency as demonstrated in columns 4 and 7 in the table.

Increased network dependency has two opposing effects - the supplier and the intermediary (firms 1 and 2) become more exposed to cash flow fluctuation which increases their default probability, but also allows for higher growth. Numerical results in Table 2.2, Column 10, show that the default probability increase dominates and the two firms merge to reduce their network exposure and default probability by creating an internal capital market which in turn increases the merger surplus of firms 1 and 2, see Column 8. The result in c) above is general for buyer-supplier networks and is due to the explicit modeling assumption where the network cash flows describe only the positive *net* effect on the supplier firms (for precise exposition see Brumen and Vanini (2006)). We assume that the buyer earns zero net profit from the buy order but can generate network cash flows from being a supplier to other firms, i.e. the supplier, contrary to the buyer, can change its suppliers without large costs. Hence the increased network dependency does not affect the supplier. The overall structure of the results depends on the parameter values chosen in Table 2.1.

Similar to the case of two firms in Section 2.2 we now consider the dependence of merger activity and firms' equity values in relation to the network average leverage ratio L. Low average leverage

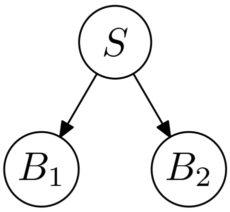
Avg. leverage ratio	Firm values with mergers			Stand-alone firm values			Merger surplus	Avg. # of coalitions	Coalitions formed
L	1	2	3	1	2	3			
2.00	2.35	24.69	117.64	2.35	24.69	117.64	0.00	3.00	(1),(2),(3)
1.60	0.06	6.28	45.52	0.06	6.28	45.52	0.00	3.00	(1),(2),(3)
1.33	8.75	5.73	15.70	3.17	0.15	10.98	15.89	1.67	(1,2,3)
1.14	25.77	20.00	14.36	9.77	2.07	0.22	48.08	1.00	(1,2,3)
1.00	35.26	27.94	19.81	18.56	9.06	3.53	51.86	1.00	(1,2,3)
0.89	43.86	35.95	27.90	28.68	19.07	14.47	45.47	1.00	(1,2,3)
0.80	52.41	35.09	37.67	39.59	30.78	29.04	25.76	1.33	(1,2,3)
0.73	52.96	45.39	50.84	50.93	43.35	44.97	9.94	1.67	(1,2,3)
0.67	62.48	56.30	61.07	62.48	56.30	61.07	0.00	3.00	(1),(2),(3)
0.62	74.10	69.33	76.82	74.10	69.33	76.82	0.00	3.00	(1),(2),(3)

Table 2.3: The dependence of the firm stock values and number of coalitions with respect to the average L ratio of the firms in the economy. The next three columns (2-4) represent the value of firms 1-3 incorporating the merger effects. Columns 5-7 are the individual firm values, not incorporating the merger effects. Column 8 gives the average number of coalitions formed. The last column shows which coalitions formed most likely, keeping in mind that for different firm bargaining orders different coalitions can arise. Except for the L parameter all other parameter values are given in Table 2.1.

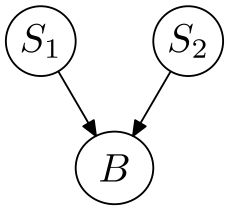
ratio L in Table 2.3, Column 1 increases the value of all firms as separate entities as well as their coalitional values, columns $5-7$ and $2-4$ respectively. Shape of merger activity resembles that in a two firm case of Section 2.2 where the majority of mergers occur for medium L values (column 9, rows $4, 5, 6$ in Table 2.3). There are no mergers for the two extreme L cases on both sides (column 9, rows $1, 2$ and $9, 10$) - no mergers occur for extremely high or low L values as indicated by the average number of 3 coalitions in column 9. Low leverage ratios virtually reduce the network dependency value thereby lowering the incentives to merge. The reason for no mergers for high levels of L comes from the asset substitution problem when the firms gamble for resurrection to increase their volatility and consequently stock values. Merger surpluses, the difference between the firm's stand-alone value and the firm value when merger activity is considered, shown in Table 2.3, column 8, increase up to a certain level (rows $1-5$ in Table 2.3) and then decrease again (rows $6-10$) as leverage ratios decrease. Merger activity (surpluses) follow an inverted (normal) U-shaped curve with respect to the average L ratio in the economy, as already pointed out in the case of two firms in Section 2.2. The difference between the network dependency and L variation in Tables 2.2 and 2.3 is that the L ratio effects all three firms in the economy by approximately equal proportions. This changes the nature of merger activity form the merger of the supplier and the intermediary (column 10, rows $2-10$ in Table 2.2) to the merger of all three firms, rows $3-8$, column 10, Table 2.3.

2.4.1 Merger effects in different types of networks

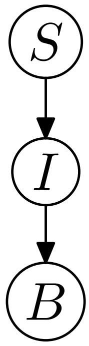
We now consider the likelihood and the type of merger, i.e. is it a merger of buyers, suppliers or a vertical merger, depending on different types of networks. In particular we consider three networks depicted in Figure 2.4. The network in Figure 2.4(a) depicts a supplier (S) to two buying firms



(a) **NT 1:** Network of multiple buyers B_1 and B_2 and a single supplier S.



(b) **NT 2:** Network of multiple suppliers S_1 and S_2 and a single buyer B.



(c) **NT 3:** Network of a buyer B, an intermediary I and a supplier S.

Figure 2.4: Different classes of buyer-supplier networks. B_i denotes the buyers, S_i the suppliers. I is the intermediary of the buyer-supplier chain in network 2.4(c).

(B_1, B_2), whereas 2.4(b) shows two suppliers (S_1, S_2) and a single buyer (B). Figure 2.4(c) shows a vertical network of a buyer, an intermediary (I) and a supplier with no relationship between the buyer (B) and the supplier (S). The adjacency matrices of networks 2.4(a), 2.4(b) and 2.4(c) are

$$\boldsymbol{E}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{E}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{E}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.12}$$

respectively. The three types of networks are distinctively different. In network 2.4(b) the default of the buyer causes a cash flow shock to both of its suppliers, whereas in 2.4(a) the default of one of the buyers causes only a cash inflow reduction. The third network 2.4(c) shares familiarities with both 2.4(a) and 2.4(b). The default of the buyer has the same network effect on the intermediary as does buyer's default in 2.4(b) on the supplier, and the default of the supplier in 2.4(c) has the same effect on the intermediary as supplier's default does on the buyer in 2.4(a). Except for the dependency matrix $\boldsymbol{E}$ all other model parameters are given in Table 2.1.

In Table 2.4 we vary the number of business relationships (links) between the individual firms by multiplying the adjacency matrix $\boldsymbol{E}_{.}$ in equation (2.12) by M. We present only the comparison of merger activity across different types of networks. Merger surplus and the average number of coalitions are defined in the same way as in Section 2.4. The complete tables with individual firm values before and after accounting for mergers for all three types of networks are presented in the Appendix, Tables 2.5, 2.6 and 2.7. The results in all three types of networks show that all three firms

	Merger surplus			Average # coalitions			Coalitions formed		
M	NT 1	NT 2	NT 3	NT 1	NT 2	NT 3	NT 1	NT 2	NT 3
1.0	79.0	0.0	0.0	2.0	3.0	3.0	(S, B_1, B_2)	$(S_1), (S_2), (B)$	(S),(I),(B)
2.0	75.6	0.0	0.0	2.0	3.0	3.0	(S, B_1, B_2)	$(S_1), (S_2), (B)$	(S),(I),(B)
3.0	112.4	0.0	0.0	2.0	3.0	3.0	$(S), (B_1, B_2)$	$(S_1), (S_2), (B)$	(S),(I),(B)
4.0	112.4	0.0	0.0	2.0	3.0	3.0	$(S), (B_1, B_2)$	$(S_1), (S_2), (B)$	(S),(I),(B)
5.0	112.4	0.0	0.0	2.0	3.0	3.0	$(S), (B_1, B_2)$	$(S_1), (S_2), (B)$	(S),(I),(B)
6.0	112.4	15.4	0.2	2.0	2.0	2.0	$(S), (B_1, B_2)$	$(S_1, S_2), (B)$	(S,I),(B)
7.0	112.4	26.3	6.8	2.0	2.0	2.0	$(S), (B_1, B_2)$	$(S_1, S_2), (B)$	(S,I),(B)
8.0	112.4	34.3	11.7	2.0	2.0	2.0	$(S), (B_1, B_2)$	$(S_1, S_2), (B)$	(S,I),(B)
9.0	112.4	40.1	15.5	2.0	2.0	2.0	$(S), (B_1, B_2)$	$(S_1, S_2), (B)$	(S,I),(B)
10.0	112.4	44.6	18.4	2.0	2.0	2.0	$(S), (B_1, B_2)$	$(S_1, S_2), (B)$	(S,I),(B)

Table 2.4: The merger surplus, number of coalitions and coalition structure for network types in Figure 2.4 and different dependency values M as multipliers of the respective adjacency matrices $\boldsymbol{E}_{.}$ in equation (2.12). NT 1 refers to the network in Figure 2.4(a), NT 2 to network in 2.4(b) and NT 3 to 2.4(c). The network dependency value has the same meaning as in Table 2.2. All other parameter values are given in Table 2.1.

never merge together (Table 2.4, Columns 8,9,10)[6] which confirms a low probability of mergers for diversification, the intuition made in Brealey and Myers (2000). All three networks also display a common feature that merger surplus increases with network dependency (columns $2, 3, 4$) which is due to the fact that for $M < 6$ the increase in firms' cash-flow volatility benefits higher firm growth, while for $M \geq 6$, columns $2, 3, 4$, firm volatility primarily increases the firms' default probability. The firms reduce their default probability through mergers. This fosters merger activity in all three types of networks. Merger surplus comparison between the network types shows that for every value of M the merger surplus in NT 1 is greater than in NT 2 which is again greater than in NT 3, columns 2, 3 and 4 in Table 2.4 respectively. This is partially explained by increased average coalition number when going from NT 1 to NT 3 in columns 5, 6 and 7 for the same value of M. We can also infer from the Table that in the case where the number of buyers (resp. suppliers) is larger than the number of suppliers (resp. buyers) (NT 1 and NT 2 resp.), mergers occur primarily in the group that dominates in the number of members.

We next analyze the merger behavior for different network types in succession. In NT 1, a network of two buyers dependent on a single supplier, the buyers merge (Table 2.4, column 8, rows $M = 3, \ldots, 9$). For NT 1 and $M = 1, 2$ the mergers oscillate between no mergers (three separate coalitions) and the coalition of all three firms, as indicated by the average number of 2 coalitions formed in column 5. The reason for the merger of the buyers is not their exposure to the supplier which does not decrease with the merger but the exposure to external cash flows, see Table 2.1.

[6] The case for NT 1 and $M = 1, 2$ is not a conglomerate merger as indicated by the average number of two coalitions in column 5.

Negative net external cash flows of buyer B_1 are balanced with positive net external cash flows of buyer B_2. Both buyers gain from the merger - the cash flows of B_1 are in average positive after the merger, whereas B_2 experiences reduction in cash flow volatility. Both buyers therefore favor the merger. The two buyers do not favor the merger with the supplier. A stagnant merger surplus for $M \geq 3$, Column 2 is a consequence of the fact that the supplier's stock is worth nearly 0 at this point.

In network type NT 2 of two suppliers and a single buyer, the merger is between the suppliers for sufficiently high dependency values (Table 2.4, column 9, rows $M = 6, \ldots, 10$). The suppliers have formed a conglomerate and effectively diversified their businesses. The merged firm of both suppliers defaults less likely than every supplier individually which generates increasing merger surplus with respect to M. As stated in the beginning of Section 2.4, it is an explicit modeling assumption that the buyers, as opposed to the suppliers, can always find an alternative supplier without greater switching costs and hence the buyer in NT 2 has no default considerations. The coalition of the suppliers does not favor joining the buyer. Its highly volatile external cash flows (Table 2.1) increase the default probability of the conglomerate more than they offset the growth rate.

Mergers in a vertical network NT 3 behave similarly as in NT 2. The merger occurs for high enough levels of network dependencies ($M \geq 6$, Column 10 in Table 2.4) between the supplier and the intermediary and does not include the buyer. We can justify this by ranking the firms by their exposure to cash-flow fluctuations. The supplier and the intermediary in network NT 3 are each exposed to its own external cash-flows as well as to the potential default of its buyers. The only exposure of the buyer is to its external cash flows. As the merger reduces the cash-flow fluctuations of the supplier and the intermediary, both firms find the merger beneficial. The buyer on the other hand has no exposure to firms higher in the buy-supply chain and faces no such considerations. The buyer hence rejects a coalition with the other firms.

2.4.2 Additional considerations and model limitations

The merger theory proposed in this Chapter relies heavily on the risk mitigation between firms and does not address several other issues important for merger decisions. Firstly, the theory proposed in this Chapter is not dynamic. Firms can merge only at the beginning of the period and stay either as separate or merged entities from then on. Using dynamic programming we can relax this assumption by allowing the firms to split and merge in response to market conditions, such as firms' asset values, volatilities, etc., to obtain a dynamic merger theory. The intuition into the behavior is given in Proposition 2.2.1 and Figure 2.3. By inverting the graph in Figure 2.3 we can view the leverage ratio L as a function of the network dependency E_{12}, i.e. $L^* = L^*(E_{12})$. When the firms' average leverage level L is below the L^* line the firms are worth their stand-alone values. When the critical L ratio is reached the firms merge. During the merger period the individual firm's value is influenced by other firms as well, see equations defining the firms' values in Proposition 2.2.1. Using the notation in Section 2.4 we can assume that the firm's optimization function is

$$S_i(t, \underline{A}) = \mathbb{E}_t \left[\int_t^{\infty} e^{-\rho_i (s-t)} LT_i(s, \underline{A}) \, \mathrm{d}s \right], \tag{2.13}$$

where ρ_i is the firm's time preference parameter, $LT_i(s, \underline{A})$ is the Leland and Toft (1996) equity value of firm i in a network with business dependencies described by $\boldsymbol{E}$ and the vector of asset values of firms in a network at time s is $\underline{A}$. The coalitions in the intertemporal framework form according to the coalition theory in Section 2.3. The stock value S_i in equation (2.13) is obviously greater than in the general model of Section 2.4 with the assumption that there are no costs to merging/spinoffs. In the setting with merger/spinoff costs the intuition suggests that there are not as many mergers as opposed to the case where these costs are absent. A reasonable way to proceed would be to assume that the coalitions form only when the asset values $\underline{A}$ reach a certain level, similar to the dynamic portfolio optimization with transaction costs in Davis and Norman (1990). Hence, there exists a range of $\underline{A}$ values where no mergers occur. Moreover, the mean-field games theory recently developed in Lasry and Lions (2006a) and Lasry and Lions (2006b) gives interacting partial differential equations which precisely describe the dynamic behavior of interacting firms.

We also assumed in this Chapter that firms gain no production efficiency as they merge, a fact refuted in for example Andrade, Mitchell, and Stafford (2001). Efficiency gains can be modelled by changing the parameters of the network model at the time of the merger or by a shock in asset value process similar to Thijssen (2008). The assumption made in this Chapter that is hardest to relax is that merged firms have no externality effects. Larger merged firms act increasingly more as monopolists which restrict competitive firm behavior assumed by the asset formation process in (2.10), as documented in Farrell and Shapiro (1990). The attempt to solve the problem of externalities was first made by Maskin (2003) but the theory developed there maintains the superadditivity axiom of the value function in coalitions and lacks the rigorous proof of its validity.

Another assumption made in this Chapter is that equity value maximization is the firms' (coalitions') principal objective and that managers always undertake merger decisions in this light, a fact ignoring the agency issues within a company. E.g., managers' incentive mechanisms, such as golden parachutes, can incite them to enter merger negotiations prematurely and thereby harm the shareholders of the company. The assumption of equity value maximization can easily be altered on the expense of computational simplicity, which would change the results, but the proposed coalitional theory in Section 2.3 is flexible enough to accomodate them. An obvious direction of further study would be the development of measures of agency conflicts and managers' incentive programs on the basis of merger behavior.

2.5 Conclusions

This dissertation chapter examines the effect of network dependencies and volatility structure of firms' cash flows on firms' equity values and the merger formation process. Merger creates an internal capital market and changes the risk-structure of cash flows that the firms generate. This creates or destroys firm value. For that purpose we develop a theory of coalition formation without the superadditivity axiom and apply it to the merger formation process. The axioms of the theory can be recursively implemented in a parallel algorithm. Firm equity value in a network should not be considered in isolation. "Merger corrections" increase the equity value of the firms accounting for the potential of a value creating merger and generate economic surplus.

In a two firm case numerical results indicate that higher network dependency increases the likelihood of a merger and merger surplus. Moreover, there exists a network dependency cutoff point above which merger is preferred by both parties. Furthermore, the dependence of merger activity with respect to the average leverage ratio of firms in an economy exhibits an inverted U-shaped function, confirming previous empirical results. The results in multi-firm networks indicate that the merger occurs predominately between either buyers or suppliers depending on which party dominates the other in number.

Possible model extensions are dynamic coalition formation with externalities.

2.A Appendix

2.A.1 Proofs of Theorems

Proof. (of Proposition 2.2.2) The proposition uses the notation and results of Section 2.3. We first assume that the firms enter the bargaining process in order $\{1, 2\}$. The only possible coalition structure after firm 1 has entered is $\mathcal{P}^1 = \{\{1\}, \emptyset\}$. Since in a network of two firms firm 2 enters last, we can compute Φ^2 for different coalitions in $\mathcal{P}^1$ (we omit the notation $\mathcal{P}^1$ from Φ^2):

$$\begin{aligned}
\Phi^2(\{1\}, \emptyset) &= S_1 \\
\Phi^2(\{1\}, \{1\}) &= S_{12} \\
\Phi^2(\emptyset, \{1\}) &= 0 \\
\Phi^2(\emptyset, \emptyset) &= S_2
\end{aligned}$$

Therefore the optimal allocation of firm 2 is to a coalition according to CA axiom is the S^o which maximizes the following

$$\begin{aligned}
S^o &= \operatorname*{argmax}_{\hat{S}\in\{\{1\},\emptyset\}} \left[\Phi^2(\{1\}, \{1\}) + \Phi^2(\emptyset, \{1\}), \Phi^2(\{1\}, \emptyset) + \Phi^2(\emptyset, \emptyset)\right] \\
&= \operatorname*{argmax}_{\hat{S}\in\{\{1\},\emptyset\}} \left[S_{12} + 0, S_1 + S_2\right]
\end{aligned}$$

Therefore $S^o(\mathcal{P}^1) = \{1\}$ if $S_1 + S_2 \leq S_{12}$ and $S^o(\mathcal{P}^1) = \emptyset$ if $S_1 + S_2 > S_{12}$. This is exactly what the condition in Proposition 2.2.2 states - if $S_{12} \geq S_1 + S_2$, firm 2 joins the coalition of firm 1, i.e. firms merge. If $S_{12} < S_1 + S_2$, firm 2 joins the coalition $\emptyset$ and operates as an independent entity. We further calculate the values φ_2 of firm 2: $\varphi_2(\mathcal{P}^1) = \Phi^2(\{1\}, \{1\}) - \Phi^2(\{1\}, \emptyset) = S_{12} - S_1$ when $S_1 + S_2 < S_{12}$ and $\varphi_2(\mathcal{P}^1) = \Phi^2(\emptyset, \emptyset) - \Phi^2(\emptyset, \{1\}) = S_2$ when $S_1 + S_2 > S_{12}$. We next compute just the needed Vickrey payments: $t_2(\{1\}|\mathcal{P}^1, \{1, 2\}) = S_{12} - S_1$ when $S_1 + S_2 < S_{12}$ and $t_2(\{1\}|\mathcal{P}^1, \{1, 2\}) = 0$ in the other case. Therefore we get ($\mathcal{P}^0 = \{\emptyset\}$) $\varphi_1(\mathcal{P}^0) = S_{12} - (S_{12} - S_1) = S_1$ and $\varphi_1(\mathcal{P}^0) = S_1 - t_2(\{1\}|\mathcal{P}^1, \{1, 2\}) = S_1$ respectively for the two cases above. By reversing the order of firms in the bargaining process, we get that the values of firms 1 and 2 are $\frac{1}{2}(S_1 + S_{12} - S_2)$ and $\frac{1}{2}(S_2 + S_{12} - S_1)$ in the case $S_1 + S_2 < S_{12}$ (exactly the Shapley values) and S_1 and S_2 in the opposite case.

□

Proof. (of Theorem 2.3.1.) To prove the theorem it suffices to show the existence (and uniqueness) of function Φ^i for every $i = 1, \ldots, N$. We show the existence part by induction on i (player number) top down, i.e. we start with player N and proceed downwards to player 1. The basis for induction is given in equation (2.5). We now assume that we have constructed Φ^i, $i = k + 1, \ldots, N$. Φ^k is then defined by equation (2.4), where t_j, $j > k$ is defined by Φ^j (which was already constructed) and $S^N(S)$ is constructed using the CA axiom. This proves the inductive step. It is obvious from the construction above that the function Φ^i is unique. □

The proof of Theorem 2.3.2 is facilitated by the series of Lemmas and the notation is adopted from there.

Lemma 2.A.1. *$\Phi^i(S, \hat{S}|\mathcal{P}^{i-1})$ is independent of $\mathcal{P}^{i-1}$ for all $i = 1, \ldots, N$.*

Proof. We prove by the downward induction on the player number k. The induction basis for player N:

$$\Phi^N(S,\hat{S}|\mathcal{P}^{N-1}) = \begin{cases} v(S) & S \neq \hat{S} \\ v(S \cup N) & S = \hat{S} \end{cases}$$

Since the coalitional game does not possess externalities, the value function v is independent of $\mathcal{P}^{N-1}$. Now we assume that all Φ^i, $i > k$ are independent of $\mathcal{P}^{i-1}$ respectively. By the CA and VP axioms and the definition (2.4) of Φ^i the induction step follows. □

Lemma 2.A.2. *Let $i+1$ be competitively allocated to S^o when the partial partition is $(\mathcal{P}^{i-1}, \hat{S} \cup i)$. Then the following relationships hold.*

(a) If $S \neq \hat{S} \in \mathcal{P}^{i-1}$ and $S^o \neq S$ then $\Phi^i(S,\hat{S}|\mathcal{P}^{i-1}) = \Phi^{i+1}(S,S^o|(\mathcal{P}^{i-1},\hat{S}\cup i))$.

(b) If $S \neq \hat{S}$ and $S^o = S$ then $\Phi^i(S,\hat{S}|\mathcal{P}^{i-1}) = \Phi^{i+1}(S,S|(\mathcal{P}^{i-1},\hat{S}\cup i)) - t_{i+1}(S|(\mathcal{P}^{i-1},\hat{S}\cup i))$.

(c) If $S \neq S^o$ then $\Phi^i(S,S|\mathcal{P}^{i-1}) = \Phi^{i+1}(S\cup i, S^o|\mathcal{P}^{i-1}, S\cup i)$.

(d) If $S = S^o$ then $\Phi^i(S,S|\mathcal{P}^{i-1}) = \Phi^{i+1}(S\cup i, S\cup i|\mathcal{P}^{i-1}, S\cup i) - t_{i+1}(S\cup i|\mathcal{P}^{i-1}, S\cup i)$.

Proof. To prove part (a) we write

$$\begin{aligned} \Phi^i(S,\hat{S}|\mathcal{P}^{i-1}) &= v(S^N(S)) - \sum_{j>i} t_j(S^{j-1}(S)|\psi^{j-1}(\mathcal{P}^{i-1},\hat{S}\cup i)) \\ &= v(S^N(S)) - \sum_{j>i+1} t_j(S^{j-1}(S)|\psi^{j-1}(\mathcal{P}^{i-1},\hat{S}\cup i, S^o\cup(i+1))) \\ &\quad - t_j(S^i(S)|\psi^i(\mathcal{P}^{i-1},\hat{S}\cup i)). \end{aligned}$$

Since $S \neq S^o$ we have that $t_j(S^i(S)|\psi^i(\mathcal{P}^{i-1},\hat{S}\cup i)) = t_j(S|\psi^i(\mathcal{P}^{i-1},\hat{S}\cup i)) = 0$, which proves part (a).

To prove part (b) we compute

$$\begin{aligned} \Phi^i(S,\hat{S}|\mathcal{P}^{i-1}) &= v(S^N(S)) - \sum_{j>i} t_j(S^{j-1}(S)|\psi^{j-1}(\mathcal{P}^{i-1},\hat{S}\cup i)) \\ &= v(S^N(S\cup(i+1))) - \sum_{j>i+1} t_j(S^{j-1}(S\cup(i+1))|\psi^{j-1}(\mathcal{P}^{i-1},\hat{S}\cup i, S\cup(i+1))) \\ &\quad - t_{i+1}(S^i(S)|\psi^i(\mathcal{P}^{i-1},\hat{S}\cup i)) \\ &= \Phi^{i+1}(S,S|\mathcal{P}^{i-1},\hat{S}\cup i) - t_{i+1}(S|\psi^i(\mathcal{P}^{i-1},\hat{S}\cup i)), \end{aligned}$$

since $t_{i+1}(S^i(S)|\psi^i(\mathcal{P}^{i-1},\hat{S}\cup i)) = t_{i+1}(S|\psi^i(\mathcal{P}^{i-1},\hat{S}\cup i))$, which proves part (b).

Part (c) and (d) are proved in a similar manner.

□

Proof. (of Theorem 2.3.2.) We fix an ordering of players entering the bargaining process as $\pi = (1,2,\dots,N)$. By averaging over all permutations $\pi \in \mathcal{S}^N$ we obtain the desired result. On occasions we will suppress certain function arguments when it is obvious from the context what they are.

To prove (a) we first establish the following identity.

$$\sum_{i\in S}\varphi_i(\mathcal{P}^{i-1}) = \Phi^k(S,S^o|\mathcal{P}^{k-1}), \tag{2.14}$$

where $S, S^0 \in \mathcal{P}^{k-1}$ and S^o is the competitive allocation (by the axiom CA) of player k. We prove (2.14) by induction on the player k getting into the bargaining process. We first establish the basis of induction.

$$\begin{aligned}\varphi_1(\{\emptyset\}) &= \Phi^1(\emptyset,\emptyset|\mathcal{P}^0) - \Phi^1(\emptyset,\emptyset'|\mathcal{P}^0)\\ &= \Phi^1(\emptyset,\emptyset|\mathcal{P}^0),\end{aligned}$$

where the identities follow from the fact that the maximum in the CW axiom (2.7) for player 1 is attained at $S^{oo} = \emptyset'$ and the fact that $\Phi^1(\emptyset,\emptyset'|\mathcal{P}^0) = 0$.

We now prove the inductive step. Let $2 \le k < N$ and $\mathcal{P}^{k-1}$ be fixed. We first assume that $S \neq S^o$ in (2.14) and differentiate between two cases: $k+1$ is competitively allocated to (i) $S^{oo} \neq S$ and (ii) $S^{oo} = S$. For the case (i) we have $\sum_{i\in S}\varphi_i(\mathcal{P}^{i-1}) = \Phi^k(S,S^o|\mathcal{P}^{k-1}) = \Phi^{k+1}(S,S^{oo}|\mathcal{P}^{k-1},S^o\cup k)$ by Lemma 2.A.2(a). This proves the inductive step in the case (i). Now consider the case (ii) $S^{oo} = S$. Here

$$\begin{aligned}\sum_{i\in S\cup(k+1)}\varphi_i(\mathcal{P}^{i-1}) &= \Phi^k(S,S^o|\mathcal{P}^{k-1}) + \varphi_{k+1}(\mathcal{P}^k)\\ &= \Phi^{k+1}(S,S|\mathcal{P}^{k-1},S^o\cup k) - t_{k+1}(S|\mathcal{P}^{k-1},S^o\cup k)\varphi_{k+1}(\mathcal{P}^{k-1},S^o\cup k)\\ &= \Phi^{k+1}(S,S|\mathcal{P}^{k-1},S^o\cup k)\end{aligned}$$

using Lemma 2.A.2 and the CW and VP axioms.

We now turn to the induction step when $S = S^o$. The case $S^{oo} \neq (S\cup k)$ is almost identical to (i) above and will not be repeated. We prove only the last case $S = S^o$ and $S^{oo} = S\cup k$. Then

$$\begin{aligned}\sum_{i\in S\cup k\cup(k+1)}\varphi_i(\mathcal{P}^{i-1}) &= \Phi^k(S,S|\mathcal{P}^{k-1}) + \varphi_{k+1}(\mathcal{P}^{k-1},S^o\cup k)\\ &= \Phi^{k+1}(S\cup k,S\cup k|\mathcal{P}^{k-1},S\cup k) - t_{k+1}(S\cup k|\mathcal{P}^{k-1},S\cup k)\\ &\quad +\varphi_{k+1}(\mathcal{P}^{k-1},S^o\cup k)\\ &= \Phi^{k+1}(S\cup k,S\cup k|\mathcal{P}^{k-1},S\cup k)\end{aligned}$$

by Lemma 2.A.2(d) and the CW and VP axioms. Altogether, this proves the induction step. Since the induction hypothesis holds also for $k = N$, we have proven (a).

Part (b) is obvious from the construction of the Vickrey payments. To prove (c) we compute

$$\begin{aligned}\varphi_k(\mathcal{P}^{k-1}) &= \Phi^k(S^o,S^o|\mathcal{P}^{k-1}) - \Phi^k(S^o,S^{oo}|\mathcal{P}^{k-1})\\ &= v(S^N(S^o\cup k)) - \sum_{j>k} t_j(S^{j-1}(S^o\cup k)|\psi^{j-1}(\mathcal{P}^{k-1},S^o\cup k))\\ &\quad -v(S^N(S^o)) + \sum_{j>k} t_j(S^{j-1}(S^o)|\psi^{j-1}(\mathcal{P}^{k-1},S^{oo}\cup k)),\end{aligned}$$

where k is competitively allocated to S^o and $S^o \neq S^{oo} \in \mathcal{P}^{k-1}$. By assumption we have that $v(S^N(S^o\cup k)) = v(S^N(S^o))$ and by Lemma 2.A.1 we have that for all $j > k$ we have $t_j(S^{j-1}(S^o\cup k)|\psi^{j-1}(\mathcal{P}^{k-1},S^o\cup k)) = t_j(S^{j-1}(S^o)|\psi^{j-1}(\mathcal{P}^{k-1},S^{oo}\cup k))$. This concludes the proof.

To prove (d) we show that the axioms we defined imply the axioms which define the Shapley value. The Pareto optimality condition of the Shapley value is implied by (a) of this theorem. The anonymity of the value function in the Shapley axioms is implied by the averaging over φ_i for different permutations. The dummy axiom is implied by (c). It remains to prove the linearity of the Shapley value, i.e. we prove that $\varphi_i^{v+v'}$ (here we make the dependence on the value function explicit) constructed from $v+v'$ equals the sum of $\varphi_i^v + \varphi_i^{v'}$. To prove this we show that $\Phi^i_{v+v'}(S_1, S_2|\mathcal{P}^{i-1}) = \Phi^i_v(S_1, S_2|\mathcal{P}^{i-1}) + \Phi^i_{v'}(S_1, S_2|\mathcal{P}^{i-1})$. This is done by induction on i. The case of Φ^N is proven from the assumption since

$$\Phi^N_{v+v'}(S_1, S_2|\mathcal{P}^{N-1}) \;=\; \begin{cases} v(S_1) + v'(S_1) & S_1 \neq S_2 \\ v(S_1 \cup N) + v'(S_1 \cup N) & S_1 = S_2 \end{cases}$$

which evidently proves the base for induction. The inductive step is a consequence of induction assumption and the equation (2.4).

□

The following Proposition demonstrates that mergers can be optimal even if synergies between certain firms are negative, thereby confirming the results in Thijssen (2008).

Proposition 2.A.3. *Let $(3, v)$ be a coalitional game of three players as in Section 2.4. We assume the following inequalities hold:*

(1) $v(\{1,2\}) < v(\{1\}) + v(\{2\})$.

(2) $v(\{1\}) + v(\{3\}) > v(\{2,3\})$

(3) $v(\{2\}) + v(\{3\}) > v(\{1,3\})$

(4) $v(\{1,2,3\}) > v(\{1,2\}) + v(\{3\})$

(5) $v(\{3\} > v(\{1\}) + v(\{2\})$

Then the axioms NA-VP predict the formation of the grand coalition even though in the game $(2, v)$ reduced to the first two players the grand coalition does not form.

Proof. We analyze the game top-down. Let us first assume that the partial partition of the first two players is $\mathcal{P}^2 = \{\{1,2\}, \emptyset\}$. Then

$$\begin{aligned}
\Phi^3(\{1,2\}, \emptyset|\mathcal{P}^2) &= v(\{1,2\}) \\
\Phi^3(\{1,2\}, \{1,2\}|\mathcal{P}^2) &= v(\{1,2,3\}) \\
\Phi^3(\emptyset, \emptyset|\mathcal{P}^2) &= v(\{3\}) \\
\Phi^3(\emptyset, \{1,2\}|\mathcal{P}^2) &= 0
\end{aligned}$$

If $\mathcal{P}^2 = \{\{1\}, \{2\}, \emptyset\}$:

$$\begin{aligned}
\Phi^3(\{1\}, \emptyset|\mathcal{P}^2) &= v(\{1\}) \\
\Phi^3(\{1\}, \{2\}|\mathcal{P}^2) &= v(\{1\}) \\
\Phi^3(\{1\}, \{1\}|\mathcal{P}^2) &= v(\{1,3\}) \\
\Phi^3(\{2\}, \{1\}|\mathcal{P}^2) &= v(\{2\}) \\
\Phi^3(\{2\}, \{\emptyset\}|\mathcal{P}^2) &= v(\{2\}) \\
\Phi^3(\{2\}, \{2\}|\mathcal{P}^2) &= v(\{2,3\}) \\
\Phi^3(\emptyset, \{1\}|\mathcal{P}^2) &= 0 \\
\Phi^3(\emptyset, \{2\}|\mathcal{P}^2) &= 0 \\
\Phi^3(\emptyset, \emptyset|\mathcal{P}^2) &= v(\{3\})
\end{aligned}$$

The optimal choice for the coalition (CA axiom) of player 3 in the case of $\mathcal{P}^2 = \{\{1,2\}, \emptyset\}$ is

$$\begin{aligned}
\max_{\{\{1,2\},\emptyset\}} \{ \; & \Phi^3(\{1,2\}, \{1,2\}|\mathcal{P}^2) + \Phi^3(\emptyset, \{1,2\}|\mathcal{P}^2), \Phi^3(\{1,2\}, \emptyset|\mathcal{P}^2) + \Phi^3(\emptyset, \emptyset|\mathcal{P}^2)\} \\
&= \max_{\{\{1,2\},\emptyset\}} \{v(\{1,2,3\}), v(\{1,2\}) + v(\{3\})\},
\end{aligned}$$

where the first element in the maximum above is connected to the coalition $\{1,2\}$ and the second to forming a new coalition ($\emptyset$).

By condition (4) above the optimal choice for player 3 in this case is the coalition $\{1,2\}$.

Let us now assume that $\mathcal{P}^2 = \{\{1\}, \{2\}, \emptyset\}$. In this case player 3 chooses between

$$\begin{aligned}
\max_{\{\{1\},\{2\},\emptyset\}} \{ \; & \Phi^3(\{1\}, \{1\}|\mathcal{P}^2) + \Phi^3(\{2\}, \{1\}|\mathcal{P}^2) + \Phi^3(\emptyset, \{1\}|\mathcal{P}^2), \\
& \Phi^3(\{1\}, \{2\}|\mathcal{P}^2) + \Phi^3(\{2\}, \{2\}|\mathcal{P}^2) + \Phi^3(\emptyset, \{2\}|\mathcal{P}^2), \\
& \Phi^3(\{1\}, \emptyset|\mathcal{P}^2) + \Phi^3(\{2\}, \emptyset|\mathcal{P}^2) + \Phi^3(\emptyset, \emptyset|\mathcal{P}^2)\} \\
= \; & \max_{\{\{1\},\{2\},\emptyset\}} \{v(\{1,3\}) + v(\{2\}) + v(\{1\}), v(\{1\}) + v(\{2,3\}) + v(\{2\}), \\
& v(\{1\}) + v(\{2\}) + v(\{3\})\}
\end{aligned}$$

By the assumptions (2) and (3) player 3 chooses its own coalition ($\emptyset$) in this case. We next analyze the competitive wages (CW axiom) for player 3 in both cases. When $\mathcal{P}^2 = \{\{1,2\}, \emptyset\}$ we have that $\varphi^3(\{\{1,2\},\emptyset\}) = v(\{1,2,3\}) - v(\{3\})$ and $t_3(\{1,2\}|\mathcal{P}^2) = v(\{1,2,3\}) - v(\{3\})$, $t_3(\emptyset|\mathcal{P}^2) = 0$. In the case when $\mathcal{P}^2 = \{\{1\}, \{2\}, \emptyset\}$ it holds that $\varphi_3(\{\{1\}, \{2\}, \emptyset\}) = v(\{3\})$.

We now proceed to the decision of player 2. The only possibility is $\mathcal{P}^1 = \{\{1\}, \emptyset\}$. We have

$$\Phi^2(\{1\}, \emptyset|\mathcal{P}^1) = v(\{1\})$$

since the optimal choice for player 3 in the case of $(\mathcal{P}^2)' = \{\{1\}, \{2\}, \emptyset\}$ is its own coalition ($\emptyset$) and

$$\begin{aligned}
\Phi^2(\{1\}, \{1\}|\mathcal{P}^1) &= v(\{1,2,3\}) - t_3(\{1,2\}) \\
&= v(\{1,2,3\}) - v(\{1,2,3\}) + v(\{3\}) \\
&= v(\{3\}),
\end{aligned}$$

since in the case of $(\mathcal{P}^2)'' = \{\{1,2\}, \emptyset\}$ the player 3 joins the coalition $\{1,2\}$. Furthermore

$$\begin{aligned}
\Phi^2(\emptyset, \{1\}|\mathcal{P}^1) &= 0 \\
\Phi^2(\emptyset, \emptyset|\mathcal{P}^1) &= v(\{2\}).
\end{aligned}$$

In the first equation we used the fact that the games ends with the grand coalition forming and the payoff to the empty coalition is 0. In the second equation that player 2 forms its own coalition and so does player 3. The payoff to this coalition is therefore $v(\{2\})$. Optimal decision for player 2 is therefore

$$\begin{aligned}\max_{\{\{1\},\emptyset\}} & \left\{ \Phi^2(\{1\},\emptyset|\mathcal{P}^1) + \Phi^2(\emptyset,\emptyset|\mathcal{P}^1), \Phi^2(\{1\},\{1\}|\mathcal{P}^1) + \Phi^2(\emptyset,\{1\}|\mathcal{P}^1)\right\} \\ &= \max_{\{\{1\},\emptyset\}}\{v(\{1\}) + v(\{2\}), v(\{3\})\}\end{aligned}$$

By assumption (5) in the Proposition player 2 chooses the coalition with player 1. It is easily seen that there exists values for v which satisfy conditions (1)-(5) above. This completes the proof. □

The following proposition is a restatement of the debt and equity pricing results in Leland and Toft (1996) and is repeated here for coherence.

Proposition 2.A.4 (Leland-Toft (1996)). *Let the dynamics of firm assets be $\frac{dA}{A} = \mu dt + \sigma dW$. The firm issued zero-coupon debt with maturity T and principal P which is retired uniformly over the interval $[0, T]$. The firm defaults when A falls below the default boundary V_B, determined below. The costs of bankruptcy are αV_B. Then the value of the equity in this model, denoted by LT (mnemonic for Leland-Toft Equity value), is*

$$LT(V, P, \sigma) = V - \alpha V_B \left(\frac{V}{V_B}\right)^{-(a+z)} - D, \tag{2.15}$$

where the value of D is given by

$$D = P\left(\frac{1-e^{-rT}}{rT} - I(T)\right) + (1-\alpha)V_B J(T), \tag{2.16}$$

where

$$\begin{aligned} I(T) &= \frac{1}{rT}(G(T) - e^{-rT}F(T)) \\ J(T) &= \frac{1}{z\sigma\sqrt{T}}\left(-\left(\frac{V}{V_B}\right)^{-a+z} N(q_1(T))q_1(T) + \left(\frac{V}{V_B}\right)^{-a-z} N(q_2(T))q_2(T)\right)\end{aligned}$$

and the constants are given by

$$\begin{aligned} F(t) &= N(h_1(t)) + \left(\frac{V}{V_B}\right)^{-2a} N(h_2(t)) \\ G(t) &= \left(\frac{V}{V_B}\right)^{-a+z} N(q_1(t)) + \left(\frac{V}{V_B}\right)^{-a-z} N(q_2(t)) \\ z &= \frac{\sqrt{(a\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}\end{aligned}$$

$$\begin{aligned} q_1(t) &= \frac{-b - z\sigma^2 t}{\sigma\sqrt{t}} & h_1(t) &= \frac{-b - a\sigma^2 t}{\sigma\sqrt{t}} & a &= \frac{r - \delta - \sigma^2/2}{\sigma^2} \\ q_2(t) &= \frac{-b + z\sigma^2 t}{\sigma\sqrt{t}} & h_2(t) &= \frac{-b + a\sigma^2 t}{\sigma\sqrt{t}} & b &= \log\left(\frac{V}{V_B}\right)\end{aligned}$$

and the default boundary V_B *is given by*

$$V_B = -\frac{AP/(rT)}{1 + \alpha(a+z) - (1-\alpha)B}$$

with

$$\begin{aligned} A &= 2ae^{-rT}N(a\sigma\sqrt{T}) - 2zN(z\sigma\sqrt{T}) - \frac{2}{\sigma\sqrt{T}}n(z\sigma\sqrt{T}) + \frac{2e^{-rT}}{\sigma\sqrt{T}}n(a\sigma\sqrt{T}) + z - a \\ B &= -\left(2z + \frac{2}{z\sigma^2 T}\right)N(z\sigma\sqrt{T}) - \frac{2}{\sigma\sqrt{T}}n(z\sigma\sqrt{T}) + z - a + \frac{1}{z\sigma^2 T} \end{aligned}$$

2.A.2 Additional tables for Chapter 2.4.1

	Firm values w/ merger effects			Firm values w/o merger effects				
M	S	B_1	B_2	S	B_1	B_2	av. # coal	Coalitions
1.0	22.9	84.6	265.1	20.8	26.4	246.4	2.0	(S, B_1, B_2)
2.0	13.2	83.0	265.1	12.8	26.4	246.4	2.0	(S, B_1, B_2)
3.0	5.8	82.6	302.6	5.8	26.4	246.4	2.0	$(S), (B_1, B_2)$
4.0	1.7	82.6	302.6	1.7	26.4	246.4	2.0	$(S), (B_1, B_2)$
5.0	0.1	82.6	302.6	0.1	26.4	246.4	2.0	$(S), (B_1, B_2)$
6.0	0.3	82.6	302.6	0.3	26.4	246.4	2.0	$(S), (B_1, B_2)$
7.0	1.6	82.6	302.6	1.6	26.4	246.4	2.0	$(S), (B_1, B_2)$
8.0	3.5	82.6	302.6	3.5	26.4	246.4	2.0	$(S), (B_1, B_2)$
9.0	5.8	82.6	302.6	5.8	26.4	246.4	2.0	$(S), (B_1, B_2)$
10.0	8.2	82.6	302.6	8.2	26.4	246.4	2.0	$(S), (B_1, B_2)$

Table 2.5: The dependence of firm stock values and the number of coalitions formed with respect to the dependency value of the network for network depicted in Figure 2.4(a). The network dependency of the whole network increases with ascending M, i.e. a network dependency value in row M is $2M$ with adjacency matrix $M \cdot \boldsymbol{E}_1$, where $\boldsymbol{E}_1$ is given in equation (2.12). The next three columns (2-4) present the value of firms 1-3 incorporating the synergy effect of the mergers. Columns 5-7 are the values of individual firms, not incorporating the merger effects. Column 8 gives the average number of coalitions formed. The last column shows which coalitions were most likely to form keeping in mind the fact that for different firm bargaining sequences different coalitional outcomes are possible. Other parameters are given in Table 2.1.

	Merger effects			No merger effects				
M	S_1	S_2	B	S_1	S_2	B	av. # coal	Coalitions
1.0	22.6	521.3	246.4	22.6	521.3	246.4	3.0	$(S_1), (S_2), (B)$
2.0	17.6	186.6	246.4	17.6	186.6	246.4	3.0	$(S_1), (S_2), (B)$
3.0	11.7	61.2	246.4	11.7	61.2	246.4	3.0	$(S_1), (S_2), (B)$
4.0	6.7	20.9	246.4	6.7	20.9	246.4	3.0	$(S_1), (S_2), (B)$
5.0	3.1	6.2	246.4	3.1	6.2	246.4	3.0	$(S_1), (S_2), (B)$
6.0	8.7	8.7	246.4	1.0	1.0	246.4	2.0	$(S_1, S_2), (B)$
7.0	13.3	13.2	246.4	0.1	0.0	246.4	2.0	$(S_1, S_2), (B)$
8.0	17.2	18.1	246.4	0.1	1.0	246.4	2.0	$(S_1, S_2), (B)$
9.0	20.8	23.0	246.4	0.7	3.0	246.4	2.0	$(S_1, S_2), (B)$
10.0	24.1	27.7	246.4	1.8	5.4	246.4	2.0	$(S_1, S_2), (B)$

Table 2.6: The dependence of firm values and the number of coalitions formed with respect to the dependency value of the network for network depicted in Figure 2.4(b). The network dependency of the whole network increases with ascending M, i.e. a network dependency value in row M is $2M$ with adjacency matrix $M\boldsymbol{E}_2$, where $\boldsymbol{E}_2$ is given in equation (2.12). The next three columns (2-4) present the value of firms 1-3 incorporating the synergy effect of the mergers. Columns 5-7 are the values of individual firms, not incorporating the merger effects. Column 8 gives the average number of coalitions formed. The last column shows which coalitions were most likely to form keeping in mind the fact that for different firm bargaining sequences different coalitional outcomes are possible. Other parameters are given in Table 2.1.

	Merger effects			No merger effects				
M	S	I	B	S	I	B	av. # coal	Coalitions
1.0	22.7	521.3	246.4	22.7	521.3	246.4	3.0	(S),(I),(B)
2.0	17.8	186.6	246.4	17.8	186.6	246.4	3.0	(S),(I),(B)
3.0	12.1	61.2	246.4	12.1	61.2	246.4	3.0	(S),(I),(B)
4.0	7.1	20.9	246.4	7.1	20.9	246.4	3.0	(S),(I),(B)
5.0	3.4	6.2	246.4	3.4	6.2	246.4	3.0	(S),(I),(B)
6.0	1.3	1.1	246.4	1.2	1.0	246.4	2.0	(S,I),(B)
7.0	3.6	3.4	246.4	0.2	0.0	246.4	2.0	(S,I),(B)
8.0	5.9	6.9	246.4	0.0	1.0	246.4	2.0	(S,I),(B)
9.0	8.3	10.7	246.4	0.5	3.0	246.4	2.0	(S,I),(B)
10.0	10.7	14.6	246.4	1.5	5.4	246.4	2.0	(S,I),(B)

Table 2.7: The dependence of firm values and the number of coalitions formed with respect to the dependency value of the network for network depicted in Figure 2.4(c). The network dependency of the whole network increases with ascending M, i.e. a network dependency matrix in row M is $2M$ with adjacency matrix $M\boldsymbol{E}_3$, where $\boldsymbol{E}_3$ is given in equation (2.12). The next three columns (2-4) present the value of firms 1-3 incorporating the synergy effect of the mergers. Columns 5-7 are the values of individual firms, not incorporating the merger effects. Column 8 gives the average number of coalitions formed. The last column shows which coalitions were most likely to form keeping in mind the fact that for different firm bargaining sequences different coalitional outcomes are possible. Other parameters are given in Table 2.1.

Chapter 3

Financial Effects of External Auditing

3.1 Introduction

The efficient market hypothesis postulates that investors possess the information to correctly value the companies in their portfolio although how this occurs is usually not examined. As Grossman and Stiglitz (1980) have demonstrated in their seminal publication on asymmetric information, the way investors acquire information is highly significant for price determination. Market participants are driven to asses costs involved in obtaining and conveying information. The empirical study by Li and Xu (2008) shows that the establishment of the 1933/34 regulatory acts which provides company information in a standardized way has reduced the volatility of financial assets at the NYSE. Auditing choice and accounting standards have been until recently overlooked in the scholarship literature. Watts and Zimmerman (1990) state that if information acquisition and transition is costless then the choice of accounting methods and auditing choice is irrelevant, p. 133. An examination of firms' actual balance sheets shows that the costs of gathering and transmitting financial information are far from negligible. For instance, the four biggest external auditing firms, PricewaterhouseCoopers, Deloitte, Ernst & Young and KPMG have posted their total global revenues in excess of 47 billion US dollars in 2007, a 12 % increase from 2006 on account of auditing services.

According to Ronen and Yaari (2007) there are two main roles of auditing: informativeness and stewardship. Informativeness, the focus of this paper, refers to the investors' demand for information in order to predict future cash flows as stated in the Statement of Financial Accounting Concept No. 1, AICPA, 1994; and AIMR, 1993. Lambert (2003) has underscored the importance of consistent summary statistics because of their ability to lessen contracting costs and transform soft information into hard one. Hard information, usually reduced to numeric form, can be transmitted easily and verifiably between economic agents, see Petersen (2004) for literature review. Past firm earnings or defaults are an example of hard information. On the other hand soft information can not be verifiably transmitted. Ideas, forecasts, future plans are all examples of soft information. Firm's stock is necessarily the combination of both hard and soft information. Auditing reduces the soft firm information part and transforms the soft into hard information, as illustrated in Chapter 5 of Petersen (2004), thereby providing a verifiable signal to the outside investors, see e.g Gibson (1999). Ronen and Yaari (2007) document that the stock market reacts differently if a firm transaction is recorded as an expense in the income statement (hard information) or simply in a footnote (soft). Our approach based on information theory brings a novel way of dealing with soft to hard information transformation on the account of auditing effort in a quantifiable manner.

We use the term “auditing” for the effort of an external auditor to verifiably convey firm cash-flow information to the outside investors. This dissertation chapter answers the following three questions:

1. What is the appropriate extent (quality) of auditing for a firm maximizing its share price?

2. Do shareholders and debtholders disagree about the optimal auditing level?

3. What are the auditor’s revenues?

We model the optimal amount of auditing services to the firms as a choice of the *auditing effort* which relates the audit costs and the precision of accounting statements, i.e. the transformation of soft into hard information. On the one hand, auditing reduces the firm’s cash flow volatility observed by outside investors while on the other hand, auditing is costly to the firm. In an economy where the representative investor is risk-averse, auditing raises stock prices. The strength of both of these effects determines the optimal auditing effort. Information theory, as developed by Shannon (1948) gives a quantitative answer as to how much and in what structural way the perceived volatility of the firm’s cash flows is reduced with the certain amount of audit effort.

Apart from auditor’s fees, Gibson (1999) identifies other indirect costs of auditing: threat of product market competition, tax avoidance considerations, agency problems among different classes of shareholders. We notice that the auditing analysis based on information theory best describes the *verification* of information and not information search. We realize that there are other functions of auditing which we do not account for. Auditing can be viewed as an extreme case of inspection games (Avenhaus, von Stangel, and Zamir (2002) and Kaplan (1993)), i.e. a disciplinary device of internal auditing. Gibson (1999) names others: enhanced auditor’s report lowers the cost of capital since they reduce the moral hazard and adverse selection problems, auditing acts as a signaling device for firms in industries where products are close substitutes, or as a reputation device.

The results show that the firm chooses the level of external auditing strictly below the level that eliminates all economic noise. If the costs of conveying the information are too high, or the noise volatility and risk-aversion are too low, the lowest auditing quality is chosen. The choice of auditing quality was empirically analyzed in Leuz and Verrecchia (2000) who show that the firms’ stock prices increase if the disclosure of information and the audit effort is *voluntarily*. The share price increase on the account of auditing has a put option like structure with respect to auditing costs and a call option like structure with respect to noise volatility, i.e. there exists a level of auditing costs (resp. noise volatility) above (resp. below) which the firm chooses the lowest audit effort which still complies with imposed auditing regulations. We prove that under certain technical conditions the auditing model is consistent with cash-flow signalling and incomplete markets arbitrage pricing models and provides simpler mathematical structure. The multi-period model of auditing preserves the one-period structural form of results for the optimal per-period auditing effort but with changed audit costs and risk aversion parameters. We prove that under general parameter conditions the level of auditing chosen by the debtholders is higher than if chosen by the shareholders, especially if the marginal auditing costs are low. Debtholders choose highest auditing levels for intermediate firm leverage values. Optimal auditing chosen by the shareholders is much smaller and generates at most a slight increase in stock price indicating that auditing primarily protects debtholders. We further show that the first best auditing contract is achieved irrespectively of whether the firm or the auditor holds the bargaining power and competition between auditors raises the auditing cutoff point. The firm’s optimization problem under incomplete information about the economic noise volatility reflects the probability that the contract will be rejected by the auditor which make a cutoff rejection/acceptance decision given its signal. Based on the results developed in previous sections, we use the Shapley value to compute the auditor's revenues and multiple units' firm stock values under auditing, a situation which proxies for both the firm size as well as for an auditor with many clients. The price obtained in this way constitutes the basis for econometric modeling.

The information theory approach to asset pricing is a relatively novel way of looking at economic agent optimization problems. One of the first papers in this area is Sims (2003) who imposes information capacity constraints on economic agents, based on the psychological results on the scarcity of attention in individuals' decision making process. The papers by Peng (2005) and Peng and Xiong (2006) extend his analysis to intertemporal financial decisions and price formation under capacity constraints. In this thesis chapter we follow the same general intuition where economic agents are firms who maximize profits with respect to the costly information constraint which reduces firm volatility. The richer framework of the information theory precisely quantifies the variance reduction in the contract theory approach to the corporate governance framework in Hermalin (2005) and Hermalin and Weisbach (2006). Leuz and Verrecchia (2000) analyzed firms following less stringent German accounting practices and show that the firms' stock prices increased for those firms in the system who disclosed additional accounting information with more audit effort *voluntarily*. Ronen and Yaari (2007) identify earnings smoothing as one of the three most familiar patterns of earnings management behavior and Lambert (1984) develops a rational principal-manager model which includes earnings smoothing and implies the reduction in earnings volatility. The results in these papers are in line with the multi-period auditing behavior in our model. The paper by Immordino and Pagano (2007) discusses the self-regulation of auditor's and starts the analysis where our paper has left off. In their paper the audit quality is unobservable and the auditor does not necessarily provide the level of auditing required by the firm. Hermalin and Weisbach (2006) report on the provision of the Sarbanes-Oxley act which requires increased reporting of off-balance sheet financing and special purpose vehicles thereby implicitly requiring the transformation of soft information about the firm profitability due to off-balance sheet assets into a much more hard information about the firm. Our model based on information theory precisely quantifies what effect does such a policy decision has on stock prices.

This chapter is structured as follows. Section 3.2 introduces the information model of optimal auditing behavior and develops the stock valuation formulas under auditing. Section 3.2.1 examines a discrete time framework of the same auditing model. In section 3.2.2 we analyze agency problems regarding auditing. Section 3.2.3 shows how the results change when competition between auditors, different bargaining power and incomplete information about the noise are introduced. Section 3.3 then develops the theory of auditing revenues for a multiple business units' firm. Section 3.4 sums up the results.

3.2 A simple model of auditing

The economic setting is as follows. We consider a firm which lives for one period and generates cash flows X at the end of that period. The firm estimates that $X \sim N(\mu, \sigma_I^2)$, i.e. the cash flows are distributed normally with mean μ and *intrinsic* volatility σ_I. The outside investors observe distorted version $\tilde{X} = X + Z$, where Z is independent of X and distributed normally with mean 0 and variance σ_N^2 which implies that $\tilde{X} \sim N(\mu, \sigma^2 = \sigma_I^2 + \sigma_N^2)$, i.e. the estimation of the cash flows by the market investors is unbiased. The *noise* component Z is non-diversifiable and influences the pricing of stocks in such economic setting, as observed by Musiela and Zariphopoulou (2008) for pricing of securities in incomplete markets. Economic noise therefore constitutes a systematic factor in this setting. The firm has a possibility to hire an external auditor. If the management chooses to do so and the auditor exerts an effort R (at a cost $c(R)$) the noise variance σ_N^2 is reduced to[1] $\sigma_N^2 \cdot 2^{-2R}$. Putting both effects together the audit effort R provides investors with cash flow distribution

$$\tilde{X}_r \sim N\left(\mu - c(R), \sigma_I^2 + \sigma_N^2 2^{-2R}\right). \tag{3.1}$$

The mechanism is depicted in Figure 3.1.

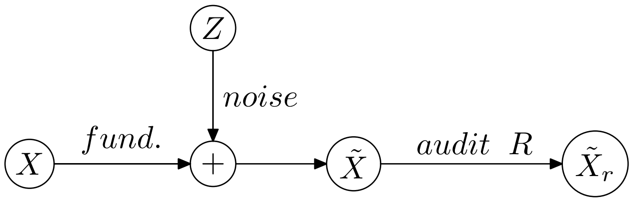


Figure 3.1: The addition of the economic noise Z to the intrinsic cash flow volatility X. The resulting cash flow is $\tilde{X}$. Auditing effort R produces firm cash flow $\tilde{X}_r$ about the firm's cash flows.

The objective function of the firm is to maximize its stock price p by choosing the audit effort R. The firm solves the following optimization problem

$$\max_{R \geq 0} \; p(\tilde{X}_r(R)). \tag{3.2}$$

There are two opposing effects of auditing. On the one side it reduces firm cash flows by the audit costs $c(R)$. On the other hand it also reduces the volatility of cash flows - risk averse investors in the financial market prefer to hold shares of low volatility firms ceteris paribus. The tradeoff between these two effects determines the optimal auditing effort.

We make the following additional assumptions. The cost function is linear[2] in R, i.e. $c(R) = C \cdot R$. The market is composed of a single representative agent with CARA utility of absolute risk aversion α. The firm's shares are in unit supply. We assume throughout this chapter that the riskless interest rate in the economy is normalized to 0. The following proposition characterizes the price formation in this setting.

Proposition 3.2.1. *In an economy described above the firm chooses auditing effort $R^* > 0$ if and only if $K = \frac{\alpha\sigma_N^2 \log(4)}{C} > 1$. The price of shares is given by*

$$p = \underbrace{\mu - CR^*}_{\mu'} - \alpha \underbrace{\left(\sigma_I^2 + \sigma_N^2 2^{-2R^*}\right)}_{(\sigma')^2}, \tag{3.3}$$

where R^* is given by

$$R^* = \begin{cases} \frac{1}{2}\log_2 K = \frac{1}{2}\left(\log_2 \alpha + \log_2 \sigma_N^2 - \log_2 C + \log_2 \log 4\right) & K > 1 \\ 0 & K \leq 1 \end{cases} \tag{3.4}$$

[1] This is the result of distortion theory for normally distributed random variables, see Appendix, Theorem 3.A.1.

[2] The structural form of the cost function assumed here does not account for the economies of scale with regards to the auditing effort. The linear costs are justified if we assume that the cost function of the auditor is in the form $CR - D\sigma_N^2$, see Proposition 3.A.9 in the Appendix, an assumption made later on. Auditing results accompanying a more general form of the cost function are presented in Proposition 3.A.4 in the Appendix. Stock pricing effects stemming form this more general setting can be treated when performing the empirical estimation.

The price of company shares has the same structure as under full information but with a changed cash flows' expectation μ' (instead of μ) and variance $(\sigma')^2$ (instead of $\sigma_I^2 + \sigma_N^2$). The CARA investors weigh the marginal costs C of establishing a credibility channel and the marginal benefits of volatility reduction $\alpha\sigma_N^2 \log(4)$. Auditing effort R^* is positively related to the coefficient of investors' risk aversion α and the noise in the economy σ_N. In the risk neutral economy, the price optimal level of auditing $R = 0$, i.e. the lowest auditing level is chosen. The investors value only expected stock returns, which are highest when the auditing level is at its minimum. Stock price (3.3) can be decomposed into the classical component $\mu - \alpha\sigma_I^2$ not influenced by auditing and a reduced economy-wide noise component $\alpha\sigma_N^2 2^{-2R^*}$ together with the auditing costs CR^*. The results for the general cost function c are given in Appendix 3.A.2, Proposition 3.A.4.

The firm's benefits from auditing B are

$$B = -CR^* + \alpha\sigma_N^2(1 - 2^{-2R^*}) = \begin{cases} \alpha\sigma_N^2\left(1 - \frac{1}{K}\right) - \frac{C}{2}\log_2 K & K > 1 \\ 0 & K \le 1 \end{cases} \tag{3.5}$$

where R^* and K are given as in Proposition 3.2.1. The auditor's revenue from auditing L is given by $L = CR^* - D\sigma_N^2$ where we assume that the auditor incurs a cost $D\sigma_N^2$ due to the auditing procedure. This costs are proportion the noise variance σ_N^2 with proportionality factor D. This assumption reflects the fact that the auditor's costs are higher in a noisier economy. We can compute the following comparative statics results ($K > 1$, $\frac{\partial B}{\partial \alpha} = \frac{\partial B}{\partial \sigma_N^2} = \frac{\partial B}{\partial C} = 0$ and $\frac{\partial L}{\partial \alpha} = \frac{\partial L}{\partial \sigma_N^2} = \frac{\partial L}{\partial C} = 0$ otherwise):

$$\begin{aligned} \frac{\partial B}{\partial \alpha} &= \sigma_N^2 - \frac{C}{2\alpha} & \frac{\partial L}{\partial \alpha} &= \frac{C}{2\alpha \log 2} > 0 \\ \frac{\partial B}{\partial \sigma_N^2} &= \alpha - \frac{C}{2\sigma_N^2} & \frac{\partial L}{\partial \sigma_N^2} &= \frac{\alpha}{K} - D = \frac{C}{\sigma_N^2 \log 4} - D \\ \frac{\partial B}{\partial C} &= \frac{1}{2} - \frac{K}{\log 4} - \frac{1}{2}\log_2 K < 0 & \frac{\partial L}{\partial C} &= \frac{1}{2\log 2}(\log K - 1) \end{aligned} \tag{3.6}$$

Graphical representation of auditing benefits and auditor's revenues from auditing is presented in Figure 3.2. The firm's auditing benefits B have a call-option like structure with respect to the overall noise in the economy σ_N^2 and a put-option like payoff with respect to the audit costs C. The intrinsic cash flow volatility σ_I does not influence firm's auditing benefits. For both parameters C and σ_N^2 there exists a cut-off value C^* and $(\sigma_N^2)^*$ such that for all $C > C^*$ and $\sigma_N^2 < (\sigma_N^2)^*$ audit benefits are 0 and the lowest level of auditing is optimal. Additionally, B becomes linear in σ_N^2 as $\sigma_N^2 \nearrow \infty$, i.e. $\frac{\partial B}{\partial \sigma_N^2} \nearrow \alpha$. The same happens for $\frac{\partial B}{\partial \alpha} \nearrow \sigma_N^2$ as $\alpha \nearrow \infty$. The payoff with respect to average risk-aversion parameter α is similar to that of σ_N^2. Similar effects are observed for L. As the risk aversion parameter α in the economy increases more auditing is demanded which increases audit fees L. The behavior of L with respect to σ_N^2 and C is non-monotonic. If the noise variance σ_N^2 is large or the audit fees C are small, the auditing revenues decrease with σ_N - the auditor has to exert large effort to reduce the noise variance to the level demanded by the firm or the collected fees do not cover the costs to the auditor. When C is low more auditing is required but the net audit effort is still low. The opposite happens when C is high.

As emphasized by Brunnermeier (2001) auditing is not the only way firms reduce their volatility. The entropy based model illuminates which signalling models are compatible with the auditing behavior described in this section. The signal we have in mind can come from various sources, such as the revealed accounting information about comparable firms, default of firm's buyers or other

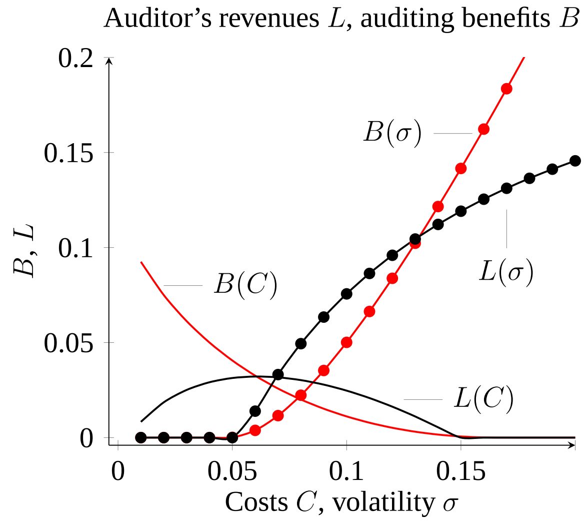


Figure 3.2: Auditor's revenues from auditing L and auditing benefits to the firm B drawn with respect to the cost of auditing C and the overall noise in the economy σ^2.

macroeconomic informations such as the economy-wide sales decrease. The investors therefore make stock price inference on the posterior distribution $X|R$, where R is the signal. In the remaining of this paragraph we show which signalling models are compatible with the structural form of posterior cash flows (3.1). We first prove the impossibility result for the jointly normal cash-flow and signal vector.

Lemma 3.2.2. *Let the firm's cash flows X be distributed normally as above and the investors receive a signal R about the economic noise Z. Then there does not exist a jointly normal distribution $(X, R)'$ such that the conditional distribution $X|R$ has structural form (3.1).*

The non-existence result shows that the signalling models consistent with the auditing behavior do not belong to the normal class. In the appendix we prove that there exist signalling models which are consistent with the structural form (3.3) where the conditional distribution $X|R$ is normal, but the signal R is not, see Appendix, Proposition 3.A.6 for details. The information theory based auditing analysis therefore restricts the set of signalling models which can be used in the analysis of auditing.

In can easily be deduced that the market model presented in this section is incomplete and there exist multiple equivalent pricing measures. Musiela and Zariphopoulou (2008) have shown that in an incomplete market the price of an asset paying out $X + Z$ with a non-tradable part Z is given by $\mathbb{E}^{\mathbb{Q}}[X + Z]$, called the *utility indifference price*, where the equivalent measure $\mathbb{Q}$ is such that the conditional distribution of $Z|(X + Z)$ is the same under both the historical measure $\mathbb{P}$ and the equivalent pricing measure $\mathbb{Q}$. We show in the Appendix, Lemma 3.A.8 that the indifference pricing in such an incomplete market coincides with the equilibrium complete market pricing under auditing possibilities, more precisely, there exists values of the auditing effort R such that the equilibrium share price of the audited firm is the same as the firm's utility indifference price.

The results in this Section are in line with the Coase theorem, where firms endogenously produce demand for auditing in order to increase share prices and raise the investors' value. The market mechanism thereby produces welfare first-best auditing scheme. The result does not obtain in an environment with debtholders, see Section 3.2.2.

3.2.1 A multi-period model

In this section we consider a firm with demand for auditing and a risk averse representative investor in a multi-period setting. The firm's cash-flows follow an AR(1) process whose mean and variance both decrease with increasing auditing effort as modelled in Section 3.2. The economic setting is similar to Vayanos (1999) with few changes to fit the auditing behavior of the firm.

Representative investor
Activity takes place at discrete times $t = 0, 1, \ldots$. There is one consumption good and two investment opportunities - a riskless one yielding a per-period return of δ and a risky investment in the stocks p of the firm. The optimization problem of the representative investor is given by

$$\max_{\{c(t),x(t)\}_{t\geq 0}} \mathbb{E}\left[-\sum_{t=0}^{\infty} \beta^t e^{-\alpha c(t)}\right] \tag{3.7}$$

subject to the wealth and equity holdings' dynamics

$$\begin{aligned} W(t) &= (1+\delta)[W(t-1) - c(t-1)] + e(t-1)X(t) - p(t)x(t-1) && (3.8) \\ e(t) &= e(t-1) + x(t-1) && (3.9) \end{aligned}$$

where $W(t)$ is wealth, $c(t)$ consumption, $e(t)$ stock holdings, $x(t)$ demand for stocks, $X(t)$ payoff from holding stocks and $p(t)$ firm's stock prices, all at time t. The parameter β is the investor's time discount parameter and α the coefficient of risk aversion. The agent is endowed with M units of consumption good at time 0.

The firm
We denote by X the cash flows to the firm. The investors observe a distorted version of X where the overall economic environment adds a noise component to X. We assume that without any auditing the cash flows perceived by the investors follow a dynamics

$$X(t) = \rho X(t-1) + S(t), \tag{3.10}$$

where $S(t)$ is a white noise process with mean μ and variance $\sigma_I^2 + \sigma_N^2$. In the presence of auditing the process S is still a white noise process but with mean $\mu(r) = \mu - Cr$ and variance $\sigma^2(r) = (1-\rho^2)(\sigma_I^2 + \sigma_N^2 2^{-2r})$, see Appendix, Proposition 3.A.3. Auditing affects both the per-period mean and standard deviation of firm's cash flows as in the one-period model in Section 3.2. The cash flows mean μ is decreased by the audit fees Cr and the per period variance is reduced to $\sigma^2(r)$. The precise value of the variance reduction is given by the information theory. The firm sets per-period auditing quality r so as to maximize the present value of the discounted future share prices p, i.e. it maximizes

$$\max_{r\geq 0} \mathbb{E}\left[\sum_{s=1}^{\infty} \gamma^s p(s)\right], \tag{3.11}$$

The supply of firm's stocks is constant and normalized to 1. It is a surprising result that *per period* level of auditing effort r coincides with the one-period auditing level R but with changed cost and risk aversion parameters. We can hence restrict the study of auditor's compensation to the one period case with multiple units.

Proposition 3.2.3. *Let the firm's optimizatio function be given by (3.11) and let $\beta\rho > \frac{1}{2}$, where β is the investor's time discount parameter and ρ is defined in (3.10). Then $r^*(t) = r^*$ is constant and has the same structural form as R^* in (3.4) with the following cost and risk-aversion parameters:*

$$C_n^1 = \frac{\rho}{2-\rho}C \tag{3.12}$$

$$\alpha_n^1 = \alpha\frac{\delta}{1+\delta}\frac{2\beta\rho-1}{\beta\rho-1}\frac{2(\rho-1)}{2-\rho} \tag{3.13}$$

A similar type of result is obtained if the optimization function of the firm is to maximize its long-run average share price. These results are presented in the Appendix, Proposition 3.A.10. The following relationship exists between the cost parameters:

$$C_n^1 < C \tag{3.14}$$

The relationship between α and α_n^1 is not that simple and depends on the relationship between β and ρ.

Ronen and Yaari (2007) on page xix identify earnings smoothing as one of the three most familiar patterns of earnings management behavior. Lambert (1984) develops a rational principal-manager model which includes earnings smoothing and implies the reduction in earnings volatility. This result is in line with Proposition 3.2.3 where the shareholders require the same degree of auditing effort per period thereby reducing the volatility of earnings to a constant amount.

3.2.2 Auditing in view of the debtholder-shareholder conflict

In this section we assume that the firm is financed with zero-coupon debt and equity and we answer the following two questions. What auditing effort would debtholders (shareholders) choose in order to maximize the value of debt (stocks)? How much are debt (stock) values increased if auditing is accounted for?

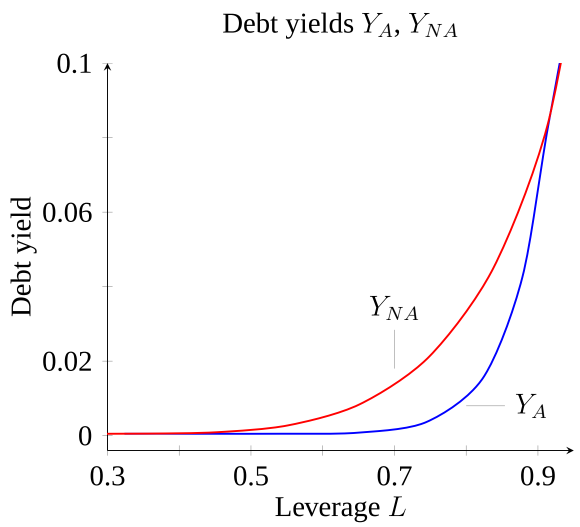
We assume the setting[3] of Leland and Toft (1996) where zero-coupon debt principal is P, its maturity is normalized to 1 and the riskless interest rate to 0. The agent (debtholder or shareholder) chooses the level of auditing effort R. By doing this, the firm's initial asset value[4] is reduced to $A_0 - c(R)$ and the firm's volatility observed by outside investors to $\left(\sigma_I^2 + \sigma_N 2^{-R}\right)^{1/2}$, as discussed in Section 3.2. The value of equity $S(R)$ and debt $D(R)$ under auditing are given in the Appendix, equations (2.15) and (2.16) respectively. Debtholders (resp. shareholders) choose the auditing effort R_D (resp. R_S) so as to maximize the value of debt (equity), i.e. $\{R_D, R_S\} = \mathrm{argmax}_{R\geq 0}\{D(R), S(R)\}$. Debt yields under the optimal auditing effort choice *for debtholders* are Y_A. Debt yields under the lowest auditing effort are denoted by Y_{NA}. It is trivial to prove that $Y_A \leq Y_{NA}$, the reason being that Y_A is the yield that maximizes the debt value (2.16). Similarly, we denote the firm's stock price under the smallest audit effort by S_{NA}. The maximal value of stock prices when the auditing decision is with the stockholders is denoted by $S_A(R_S)$ and by $S_A(R_D)$ if

[3] Almost the same results are obtained in a Merton (1974) model. We present those results in the Appendix, Section 3.A.3.

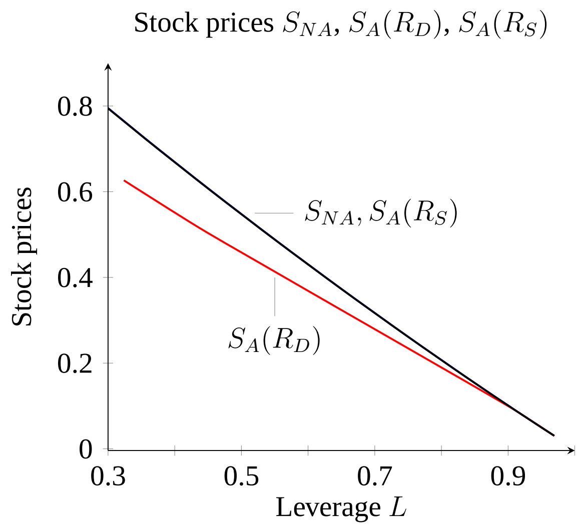
[4] Under certain technical conditions, see Dixit and Pindyck (1994) or Proposition 1.2.1, the firm's cash flows and asset values are proportional and we can assume that the auditing costs affect both in equal amount.

the decision is with the debtholders. The difference $S_A(R_S) - S_A(R_D)$ is a measure of agency costs originating from the decision about the auditing procedure.

Figure 3.3 shows the relationship between firm leverage L on the horizontal axis and debt yields (left, Figure 3.3(a)) / stock prices (right, Figure 3.3(b)) on the vertical one. The optimal auditing



(a) Debt yields under auditing Y_A and under lowest auditing effort procedure Y_{NA}.



(b) Stock prices under the lowest audit effort S_{NA}, when auditing effort is chosen by the debtholders $S_A(R_D)$ and when the auditing is chosen by the shareholders $S_A(R_S)$.

Figure 3.3: The relationship between firm leverage L and debt yields Y (equity prices S) with auditing and under the lowest audit regime. Model parameters are as follows: auditing costs $C = 0.1$, intrinsic cash flow volatility $\sigma_I = 0.15$, noise level volatility $\sigma_N = 0.3$ and initial firm asset value $A_0 = 1.2$.

effort R that the shareholders/debtholders require is different. Debtholders require most auditing protection at intermediate leverage levels (Figure 3.3(a)) and much less for extremely low and high leverage where the audit and non-audit curves almost coincide. Low leverage levels imply very low debt riskiness and little if any auditing protection is needed. The situation is reversed in a high leverage situation where debt riskiness is amplifyed by additional auditing costs. It follows from the considerations above that the “total costs of auditing”, that is the sum of direct costs and the costs of increased default probability, exhibit an inverted bell shaped form from the perspective of debtholders. A very different picture emerges when we look at auditing effects on the shareholder value, Figure 3.3(b). Curves depicting stock values with lowest auditing effort possible S_{NA} and with auditing chosen optimally by the shareholders $S_A(R_S)$ almost coincide[5]. Without any restriction on audit requirements the shareholders normally choose the lowest auditing level possible. If the auditing is chosen optimally by the bondholders (lower line in Figure 3.3(b)) the share value decreases and the shareholders are made worse off at every leverage level. Debt yield (stock price) figures scale with different parameter values of C, σ_I, σ_N but the shape remains pretty stable. The only cases when $S_A(R_S) > S_{NA}$ is for extremely small values of the intrinsic volatility parameter ($\sigma_I \approx 3\%$ as opposed to $\sigma_I = 15\%$ considered in Figure 3.3) and very small audit cost parameter C and even for these cases the audited stock price $S_A(R_S)$ is somewhat higher than S_{NA} only for

[5]The parameter values can be chosen so that the stock value under auditing can be higher than when the lowest auditing level is selected but the cost function has to be unrealistically orders of magnitude smaller than other parameters. Even for those parameter values the auditing surplus is extremely small.

certain leverage values L. We perform the robustness control for auditing cost function (which we assumed linear in auditing effort CR as in Section 3.2) in the empirical section where a more general cost function is assumed. The auditing effects in the Merton (1974) model mentioned above yield almost exactly the same results and are presented in Appendix 3.A.3.

3.2.3 Strategic considerations regarding auditing

In the previous sections we assumed that both intrinsic and noise volatilities σ_I, σ_N are known, that both the firm and the auditor agree on the extent of auditing and that there is only one auditor. In this section we relax these assumptions and examine the auditing decision under these new conditions.

Varying the degree of bargaining power

We first address the issue of negotiating power, that is which party offers the contract. We distinguish between two extreme cases. In the first one, the firm offers an auditing contract with specific auditing quality (effort) level. A single auditor can then either accept or decline the contract. All other inputs and models assumptions, such as investor's preferences and parameters, are the same as in Section 3.2.

Proposition 3.2.4. *Let R^* be the first best auditing effort as defined in (3.4), Proposition 3.2.1. If the firm offers the auditing contract and a single auditor can only accept or decline it, then the first best auditing effort R^* is accepted if and only if*

$$\log \frac{\alpha \sigma_N^2 \log 4}{C} > \frac{2D\sigma_N^2}{C}. \tag{3.15}$$

If the condition (3.15) fails the audit contract is not accepted by the auditor.

In the second example, we consider two auditors, which simultaneously submit audit offers. The firm then chooses the one closest to its optimal audit level. If both auditors submit the same auditing offer, they share the auditing profits equally. The following proposition shows that in favorable economic condition (specified precisely below) the first best auditing level is always achieved, i.e. the same contract is proposed when the firm or the auditor(s) offer the contract. The difference between the two cases emerges when the overall economic conditions are not sufficiently favorable.

Proposition 3.2.5. *Let R^* be the first best auditing effort as defined in (3.4), Proposition 3.2.1. If two auditors simultaneously submit audit offers, than the optimal auditing effort is R^* if and only if*

$$\log \frac{\alpha \sigma_N^2 \log 4}{C} > \frac{4D\sigma_N^2}{C}. \tag{3.16}$$

If the condition (3.16) fails the audit contract is not accepted by the auditor.

Conditions (3.15) and (3.16) are the participation constraints on the auditor's side. Results in Propositions 3.2.4 and 3.2.5 show that even though the auditor and the firm have different optimizing functions *only* the first best contract from the *firm's* perspective is always chosen. This is due to the fact that the auditor's profit function is increasing with audit effort R while the firm's profits are decreasing after a certain cutoff value of the auditing level. Even though the auditor's profits would increase after the first best auditing level, the firm rejects the auditing contract. In cases when auditing is beneficial for both parties, that is when conditions (3.15) and (3.16) are satisfied, the firm maximizes its profit due to auditing and the auditor accepts it. The difference between the two cases is what are the economic conditions that allow it. We analyze only condition (3.15), the other one follows exactly the same reasoning. Not surprisingly, the audit contract is accepted if the coefficient of risk aversion α is high enough. The decrease of cost parameter C has an ambiguous role. Its decrease means lower auditing costs to the firm but at the same time lower profit to the auditor. As $C \searrow 0$ the auditing becomes non-profitable for the auditor and as $C \nearrow \infty$ the auditing becomes unprofitable for the firm. It is the medium range of C values where the contract if profitable for both parties. Economic noise σ_N has almost the exact opposite effect to C. In an environment where economic noise is large the demand for auditing is high but the costs to the auditor are also large. When the noise is small firms usually do not demand auditing. The auditor's cost D act only as a cut-off point. High D moves the auditing cut-off to the left while low values of D move the cutoff audit level R to the right. The condition (3.16) is stricter than (3.15), that is the case when the contract is offered by the auditors is accepted less often as when offered by the firm which is due to the competition on the auditors' side, i.e. the profits from auditing have to be shared among the auditors which decreases the likelihood of a contract being accepted. Similarly to the case of debtholder/shareholder conflict there exist room for welfare improvement due to limiting competition in the auditors' market. This line of reasoning of course ignores many other frictions regarding auditors, such as technological innovation in monitoring and gathering information, etc.

The case of unknown σ_N

In Section 3.2 we assumed that the noise variance parameter σ_N was known by both the firm and the auditor perfectly. Here we relax this by imposing that only the firm, but not the auditor has perfect knowledge about σ_N^2. We model this behavior in a framework of signalling games, see Fudenberg and Tirole (1991), Chapter 3. There exist two players, the shareholders of the firm (called only the firm), player 1 and the auditor, player 2. All variables in the remaining part of this section with index 1 and 2 refer respectively to the firm and the auditor. The game proceeds in the following fashion. The firm proposes a contract of auditing effort level r to the auditor taking into account the auditor's uncertainty about σ_N^2. The auditor receives independently of the firm a signal S_2 regarding σ_N^2 and decides whether to accept or decline the audit offer. If the auditor declines, it earns zero profit. If the auditor accepts the auditing contract, it collects the audit fee and incurs a cost $D\sigma_N^2$, as described below equation (3.5) on page 65. The auditor's response function $s_2(S_2)$ therefore takes only values in $\{\text{Accept}, \text{Decline}\}$.

The signal S_2 of the auditor can come from a multitude of sources. The auditors usually audit a large portfolio of firms and have knowledge of noise amount about the industry sector to which the firm belongs. Furthermore, the auditors can compare firm share prices to those of other similar ones and obtain more or less reliable information about the noise level. The firm on the other hand knows σ_I (and therefore σ_N) perfectly. The game matrix for this signalling game is given in Table 3.1. The following proposition characterizes the actions of both the firm and the auditor in this setting.

Proposition 3.2.6. *Let $\mathbb{E}(\sigma_N^2|S_2) = S_2$. In the setting above the firm proposes a contract of auditing level r^* which maximizes*

$$r^* \in \operatorname*{argmax}_r \left\{ N(b')(-Cr + \alpha\sigma_N^2(1 - 2^{-2r})) \right\}, \tag{3.17}$$

where $b'(r) = \frac{Cr}{D}$ and $N(x) = \mathbb{P}(S_2 \leq x)$ is the distribution function of S_2. The auditor's decision

		Auditor	
		Accept	Decline
Firm	r	$(\mu - Cr - \alpha\sigma_I^2 - \alpha\sigma_N^2 2^{-2r}, Cr - D\sigma_N^2))$	$(\mu - \alpha\sigma_I^2 - \alpha\sigma_N^2, 0)$

Table 3.1: The game matrix for the signalling game between the firm and the auditor. The first entry in the parenthesis is the payoff to the firm and the second to the auditor for both actions (Accept or Decline) of the auditor. The allowed values for $r \in \mathbb{R}_+$.

s_2 to accept the offer has a cutoff value $b'(r)$ with signals S_2 below $b'(r)$ eliciting an acceptance of the auditing procedure, i.e.

$$s_2^*(S_2) = \begin{cases} \textit{Accept} & S_2 \leq b'(r) \\ \textit{Decline} & S_2 > b'(r) \end{cases}$$

The condition that $\mathbb{E}(\sigma_N^2|S_2) = S_2$ is plausible in a normal setting, i.e. if (σ_N^2, S_2) are jointly normal and S_2 is an unbiased statistic of σ_N^2. The decision of the auditor is driven by its profit meaning that it accepts audit offers when the signal S_2 about the noise level volatility σ_N^2 compared to σ_N^2 is low enough, implying low enough costs of auditing. The conditions of this sort are very common in the literature on global games and other games of imperfect information.

The comparison between the firm's/auditor's decision making in the complete and incomplete information setting is presented in Table 3.2. The major difference for the auditor is the replacement

	Complete information	Incomplete information
Auditor's cutoff point	$\frac{D\sigma_N^2}{C}$	$\frac{DS_2}{C}$
Firm's opimizing function	$-CR - \alpha\sigma_N^2 2^{-2R}$	$N(b')(-Cr + \alpha\sigma_N^2(1 - 2^{-2r}))$

Table 3.2: The comparison of auditor's cutoff point and firm's optimization function in the complete and incomplete information setting.

of the noise variance parameter σ_N^2 in the complete information case with the signal S_2 about σ_N^2 in the incomplete setting. Regarding the firm's optimization function (Table 3.2, second row) we observe that the complete information auditing costs C are lowered to $CN(b')$ in the incomplete information case and that the risk-aversion parameter α of the complete information setting is replaced by $\alpha N(b'(r))$. Since r enters non-linearly in the firm's optimization function the direction of audit effort change in comparison to the complete information case is ambiguous. We show in Proposition 3.A.12 in the Appendix that the incomplete information case converges to the complete information case in Section 3.2 as the parameter uncertainty about the economic noise σ_N^2 decreases to zero, i.e. the firm's objective functions (3.17) and (3.2) coincide.

3.3 A model of auditor's revenues

We now apply the theory developed in Section 3.2 to the revenues of external auditors. We assume a stylized one-period model where a firm is composed of n business units, each of which is a source of cash flows with varying riskiness as described in Section 3.2. Subsection 3.2.1 confirmed that the analysis of a multi-period case is essentially the same as the one-period. Firm's business units proxy for both firm size as well as the number of audited firms by a single auditor. The firm has the possibility to invest resources into auditing which reduces the overall cash flow volatility of each individual business unit as perceived by outside investors.

There are two types of agents in the model - an external auditor, which we denote by a and n individual firm business units. Each business unit i ($i = 1, \dots, n$) generates at the end of the period a cash flow $X_i + Z_i$ to the firm where X_i and Z_i are independent, $X_i \sim N(\mu_i, \sigma_{Ii}^2)$ and $Z_i \sim N(0, \sigma_{Ni}^2)$. The firm has a choice of employing an external auditor. If it does so the auditing costs are Cr per business unit and the volatility of the business unit is reduced from $\sigma_{Ii}^2 + \sigma_{Ni}^2$ to $\sigma_{Ii}^2 + \sigma_{Ni}^2 2^{-2r^*}$. Firm's stock pricing is done the same way as in Section 3.2. We denote by $v(A)$ the value of a certain subset of firm's business units (A subset of $\{1, \dots, n\}$) under the lowest auditing level scenario and $v(a, A)$ the value of these units under external auditing. The following relationships hold:

$$
\begin{aligned}
v(\emptyset) = v(a) &= 0 \\
v(A) &= \sum_{i \in A} \left(\mu_i - \alpha\sigma_{Ii}^2 - \alpha\sigma_{Ni}^2\right) &\quad (3.18) \\
v(a, A) &= \left(\sum_{i \in A}(\mu_i - Cr^* - \alpha\sigma_{Ii}^2) - \alpha|A| \prod_{i \in A} \frac{\sigma_{Ni}^{2/|A|}}{2^{2r^*}}\right) &\quad (3.19)
\end{aligned}
$$

where the notation is the same as in Section 3.2. Since business units are assumed to be independent[6] the value of a subset A of firm's business units under the lowest auditing effort (3.18) is the sum of individual business unit contributions to the total share price as in Section 3.2, equation (3.3). The volatility of a set of business units when the auditor exerts an effort r^* is reduced according to Proposition 3.A.2, result (3.25) in Appendix 3.A.2). We use Shapley value as a solution concept to determine the auditor's revenues and the value added by the auditor to the firm with n business units.

Proposition 3.3.1. *Under the assumptions stated above the auditor's revenue φ_a of auditing a firm with n business units described above is*

$$\varphi_a = \alpha\frac{n-1}{n}\sum_{i=1}^{n}\sigma_{Ni}^2 - \alpha \sum_{S \subset \{1,\dots,n\}} \frac{s!(n-s-1)!}{n!} s \prod_{i \in S} \frac{\sigma_{Ni}^{2/s}}{2^{2r^*}} - Cr^* n(H(n) - 1) \quad (3.20)$$

where

$$r^* = \frac{1}{2n}\log\frac{2n(\log 2)\alpha\left(\prod_{i=1}^{n}\sigma_{Ni}^2\right)^{1/n}}{C} \quad (3.21)$$

and $H(n)$ is the n-th harmonic number. The total market firm capitalization is then

$$
\begin{aligned}
v(a, \{1, \dots, n\}) = & \underbrace{\sum_{i=1}^{n}(\mu_i - \alpha\sigma_{Ii}^2)}_{T_1} \underbrace{- \alpha \sum_{S \subset \{1,\dots,n\}} \frac{s!(n-s-1)!}{n!} s \prod_{i \in S} \frac{\sigma_{Ni}^{2/s}}{2^{2r^*}}}_{T_2} \\
& \underbrace{- \frac{C}{2}\log\frac{2n(\log 2)\alpha\left(\prod_{i=1}^{n}\sigma_{Ni}^2\right)^{1/n}}{C}}_{T_3} \quad (3.22)
\end{aligned}
$$

[6] The independence assumption can be removed with significant mathematical difficulties but does not offer any new economic insights. In the framework of correlated business units generating normally distributed cash flows, the firm's business units can be redefined so as to become independent.

The equation (3.21) equates the marginal costs of auditing $Cn(H(n)-1)$ with the marginal benefits (second term in (3.20)). The marginal costs of a unit of auditing increases with the number of firm's business units almost linearly since $\lim_{n\to\infty}\frac{H(n)}{n}=0$ where the proportionality factor is the auditing costs C. The marginal benefit of auditing is a decrease in overall firm's volatility as indicated by the second term of (3.20). The marginal benefits depend positively on the level of risk aversion parameter α and in a non-linear fashion on the volatilities of firm's business units. Equation (3.20) reveals that the size of the auditor's contribution to the firm value does not depend on the mean production level μ of firm's business units. The value of the firm under auditing (3.22) is the standard valuation without economic noise (term T_1 in 3.22), reduced by the now decreased noise volatility T_2 and the optimal cost of auditing T_3.

3.4 Conclusions

This chapter develops a model of optimal auditing behavior when cash flows to the firm are observed imperfectly by the outside investors. An external auditor's report produces a verifiable signal and reduces the observed cash flow volatility. Using the results in information theory we develop explicit formulas for firm's share price under auditing in the Gaussian-CARA framework in which the firm's auditing benefits have a call-option like structure with respect to the noise volatility and a put-option like payoff with respect to the marginal audit costs. The multi-period model of auditing preserves the one-period structural form of results for the optimal per-period auditing effort but with changed audit costs and risk aversion parameters. The shareholders and debtholders in a firm disagree about the optimal auditing effort which primarily shields debtholders. Under sufficiently favorable economic conditions we obtain that the first best audit contract is offered irrespective of the bargaining power between the auditor and the firm and the number of auditors competing for the same contract. The auditor makes a cut-off decision with respect to the signal about economic noise when the noise volatility is not observed perfectly. Finally we develop the auditing profits and firm values for a multi-unit firm which lays ground for empirical testing.

3.A Appendix

3.A.1 Basic results of Information Theory

Theorem 3.A.1 (adapted from Theorem 13.3.2 in Cover and Thomas (1991)). *In a noisy Gaussian channel where the input variable is distributed normally with mean zero and variance σ^2, the size R of the channel necessary to reduce the variance of the input to D is*

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D} & 0 < D \leq \sigma^2 \\ 0 & D > \sigma^2 \end{cases} \tag{3.23}$$

We call R also the *rate distortion* parameter. Theorem 3.A.1 gives the size of the channel R if all normally distributed random variables with variance D or less are to be communicated without error. By inverting (3.23) and solving for D we obtain that the normally distributed random variables with variance $D = \sigma^2 2^{-2R}$ can be communicated perfectly over the channel of size R.

The next result establishes the link between channel size and total variance reduction for normally distributed random vectors.

Theorem 3.A.2 (adapted from Theorem 13.3.3 in Cover and Thomas (1991)). *In a noisy Gaussian channel for a vector $\underline{X}$ of dimension k of independently distributed random variables with variances $\{\sigma_i^2\}_{i=1,\ldots,k}$ respectively, the size of the channel necessary to reduce the total variance of the input to D is*

$$R(D) = \sum_{i=1}^{k} \frac{1}{2}\log\frac{\sigma_i^2}{D_i}, \tag{3.24}$$

where $D_i = \min(\lambda, \sigma_i^2)$ and $\sum_{i=1}^{k} D_i = D$.

In case when $D < k\min\{\sigma_1^2, \ldots, \sigma_k^2\}$ we have from (3.24) that

$$D = k\prod_{i=1}^{k}\left(\frac{\sigma_i^2}{e^{-2R(D)}}\right)^{1/k} \tag{3.25}$$

The last result concerns rate distortion of a sequence of random variables following an AR(1) serially correlated process.

Proposition 3.A.3. *Assume that $R \geq \log(1+\rho)$. The rate distortion of a Gauss-Markov source process $X(t) = \rho X(t-1) + S(t)$, where $0 < \rho < 1$ and S is an i.i.d. Gaussian $N(0,\sigma^2)$ sequence is given by*

$$D(R) = (1-\rho^2)\sigma^2 2^{-2R}.$$

3.A.2 Proofs of Theorems

Proof. (of Proposition 3.2.1.) Since $\lim_{R\to\infty}\sigma^2(R) = \sigma_I^2$ and $\lim_{R\to 0} c(R) = 0$ we have that the maximum is achieved either when $R = 0$ or in the interior of the positive real axis, when $\frac{\partial p}{\partial R} = 0$. The first order condition for this is

$$\frac{\partial p}{\partial R} = -C - \alpha\sigma_N^2 2^{-2R}\log(2^{-2})$$

which establishes (3.4). If $R^* > 0$ then

$$\frac{\partial^2 p}{\partial R^2} = -\alpha\sigma_N^2 2^{-2R^*}\left(\log\frac{1}{4}\right)^2 < 0,$$

which guarantees that $p(R^*)$ is indeed a maximum. □

Proposition 3.A.4. *In an economy of Section 3.2, the firm chooses auditing effort $R^* > 0$ if and only if there exists a positive solution to*

$$c'(R) = \alpha\sigma_N^2(\log 4)2^{-2R}. \tag{3.26}$$

The price of the shares is given by

$$p = \underbrace{\mu - c(R^*)}_{\mu'} - \alpha\underbrace{\left(\sigma_I^2 - \alpha\sigma_N^2 2^{-2R^*}\right)}_{(\sigma')^2}. \tag{3.27}$$

The proof of Proposition 3.A.4 follows the same lines as that of Proposition 3.2.1 and will be omitted.

The following Proposition proves that the audit costs are linear (affine) provided that the costs of the auditor are linear in its effort.

Proposition 3.A.5. *Let the assumptions of Section 3.2 hold with the exception of the structural form of c and let the unknown variable $\sigma_N \in \{\sigma_N^H, \sigma_N^L\}$ be known to the auditor but unknown to the firm, see the discussion in Section 3.2.3. Then the audit costs C_H, C_L connected to different levels of σ_N are given by*

$$C_H = D\sigma_N^H R_H \tag{3.28}$$
$$C_L = D\sigma_N^L R_L + D(\sigma_N^H - \sigma_N^L)R_H. \tag{3.29}$$

Proof. We formulate the firm's contracting problem as

$$\max_{C_H, C_L, R_H, R_L} \mathbb{E}\left[\mu - C_i - \alpha(\sigma_I^2 + \sigma_N^{i2}2^{-2R_i})\right] \tag{3.30}$$

subject to the auditor's incentive compatibility constratints

$$C_H - D\sigma_N^H R_H \geq C_L - D\sigma_N^H R_L \tag{3.31}$$
$$C_L - D\sigma_N^L R_L \geq C_H - D\sigma_N^L R_H \tag{3.32}$$

and participation constraints

$$C_H - D\sigma_N^H R_H \geq 0 \quad (3.33)$$
$$C_L - D\sigma_N^L R_L \geq 0 \quad (3.34)$$

It is easy to show that

$$C_L - D\sigma_N^L R_L \geq C_H - D\sigma_N^L R_H \geq C_H - D\sigma_N^H R_H \geq 0 \quad (3.35)$$

where the middle inequality is inferred from the incentive compatibility constraint (3.51), the second one since $\sigma_N^H \geq \sigma_N^L$ and the third one from the participation constraint (3.52). Therefore (3.52) holds with equality and we arrive at (3.47). Using (3.47) in (3.51) we get that

$$C_L - D\sigma_N^L R_L \geq C_H - D\sigma_N^L R_H = D(\sigma_N^H - \sigma_N^L)R_H > 0$$

and therefore we can not have $C_L - D\sigma_N^L R_L = 0$ but rather (3.48), an affine function of R_L. It therefore follows that the cost function is either linear or affine, not very differnt than the assumed $c(R)$. □

Proposition 3.A.6. *There exist independent random variables S and a standard normal random variable Z such that X given by*

$$X = \mu - CS + Z\sqrt{\sigma_I^2 + \sigma_N^2 2^{-2S}}.$$

satisfies

$$\mathbb{E}(X) = \mu \quad (3.36)$$
$$\textit{var}(X) = \sigma_I^2 + \sigma_N^2 \quad (3.37)$$
$$\mathbb{E}(X|S) = \mu - CS \quad (3.38)$$
$$\textit{var}(X|S) = \sigma_I^2 + \sigma_N^2 2^{-2S} \quad (3.39)$$

Proof. The results for the conditional mean and variance (3.38), (3.39) as well as the unconditional mean (3.36) are obvious. It remains to calculate the conditional variance (3.37). For that purpose we first compute:

$$\begin{aligned}\mathbb{E}(X^2) &= \mu^2 + C^2\mathbb{E}(S^2) + \mathbb{E}(Z^2(\sigma_I^2 + \sigma_N^2 2^{-2S})) \\ &= \mu^2 + C^2\mathbb{E}(S^2) + \mathbb{E}(\sigma_I^2 + \sigma_N^2 2^{-2S})\end{aligned}$$

In order to obtain the desired result the following identity has to hold

$$\frac{C^2}{\sigma_N^2}\mathbb{E}(S^2) = 1 - \mathbb{E}(2^{-2S}), \quad (3.40)$$

subject to $\mathbb{E}(S) = 0$. Let M be the moment generating function of S. We write the Taylor expansion of M as

$$M(t) = 1 + v\frac{t^2}{2} + O(t^3),$$

where v is the variance of S. For (3.40) to hold the following equation has to be satisfied

$$av = -\frac{(2\log 2)^2}{2}v - O((-2\log 2)^3),$$

where $a = \frac{C^2}{\sigma_N^2}$ and $\mathbb{E}(2^{-2S}) = M(-2\log 2)$. The condition for the existence of solution to (3.40) is that the Taylor series expansion of the moment generating function M of S from third term on evaluated at $-2\log 2$ is positive: $-O((-2\log 2)^3) > 0$. We prove in Proposition 3.A.7 that there does not exist a normally distributed random variable S such that (3.40) holds except when $S = 0$ a.s. $\square$

Proposition 3.A.7. *There does not exist a joint normal distribution (X, S) such that (3.40) holds.*

The Proposition is equivalent to solving the equation

$$av = 1 - e^{2v\log^2 2}$$

where $a = \frac{C}{\sigma_N^2}$ which has a solution only for $v = 0$, which implies that $S = 0$ a.s.

Proof. (of Lemma 3.2.2.) Let $R \sim N(0, \sigma_R^2)$ correlated with cash flows $X \sim N(\mu, \sigma_I^2 + \sigma_N^2)$ where the correlation coefficient is ρ. The conditional distribution $X|R \sim N(\mu_{X|R}, \sigma_{X|R}^2)$ where

$$\mu_{X|R} = \mu + R\frac{\rho\sqrt{\sigma_I^2 + \sigma_N^2}}{\sigma_R} \quad (3.41)$$

$$\sigma_{X|R}^2 = (1-\rho)^2(\sigma_I^2 + \sigma_N^2) \quad (3.42)$$

But (3.42) is not consistent with the variance decrease given in (3.1) which implies that

$$\sigma_{X|R}^2 = \sigma_I^2 + \sigma_N^2 2^{-2R}$$

or in other words that

$$\rho = 1 - \sqrt{\frac{\sigma_I^2 + \sigma_N^2 2^{-2R}}{\sigma_I^2 + \sigma_N^2}},$$

a function of R. $\square$

Lemma 3.A.8. *Let the vector $(X, Z)'$ be distributed normally with mean $(\mu, 0)'$ and variance-covariance matrix $\begin{bmatrix} \sigma_I^2 & 0 \\ 0 & \sigma_N^2 \end{bmatrix}$ and let the equivalent measure changes be such that the vector $(X, Z)'$ under the new measure is also distributed normally with mean $(\mu_1, \mu_2)'$ and variance-covariance matrix $\begin{bmatrix} (\sigma_I^{\mathbb{Q}})^2 & 2\rho\sigma_I^{\mathbb{Q}}\sigma_N^{Q} \\ 2\rho\sigma_I^{\mathbb{Q}}\sigma_N^{Q} & (\sigma_N^{\mathbb{Q}})^2 \end{bmatrix}$. Then in order that the conditional distribution $Z|X+Z$ is the same under both measures the following must hold:*

$$\frac{(\sigma_N^{\mathbb{Q}})^2 + \rho\sigma_N^{\mathbb{Q}}\sigma_I^{\mathbb{Q}}}{V} = \frac{\sigma_N^2}{\sigma_I^2 + \sigma_N^2} := A \quad (3.43)$$

$$\frac{(1-\rho^2)(\sigma_N^{\mathbb{Q}})^2(\sigma_I^{\mathbb{Q}})^2}{V} = \frac{\sigma_I^2\sigma_N^2}{\sigma_I^2 + \sigma_N^2} =: AB \quad (3.44)$$

$$\mu A = \mu_2 - (\mu_1 + \mu_2)\frac{(\sigma_N^{\mathbb{Q}})^2 + \rho\sigma_N^{\mathbb{Q}}\sigma_I^{\mathbb{Q}}}{V} \quad (3.45)$$

where $V = (\sigma_I^Q)^2 + (\sigma_N^{\mathbb{Q}})^2 + 2\rho\sigma_I^{\mathbb{Q}}\sigma_N^{\mathbb{Q}}$*. The solution to (3.43)-(3.45) exists and the distribution of* $X + Z$ *under* $\mathbb{Q}$ *is normal with mean* $\mu_2 \frac{\sigma_N^2+\sigma_I^2}{\sigma_N^2} - \mu$ *and variance* V*.*

Proof. The distribution of $Z|X+Z$ under both measures is normal. It therefore suffices to compute conditional mean and variance. Under $\mathbb{P}$ the following holds:

$$\begin{aligned}
\mathbb{E}(Z|(X+Z)=x) &= (x-\mu)\frac{\sigma_N^2}{\sigma_I^2+\sigma_N^2} \\
\text{var}(Z|(X+Z)=x) &= \sigma_N^2 - \frac{\sigma_N^4}{\sigma_I^2+\sigma_N^2} = \frac{\sigma_I^2\sigma_N^2}{\sigma_I^2+\sigma_N^2}.
\end{aligned}$$

Under the equivalent measure $\mathbb{Q}$ the following relationships hold:

$$\begin{aligned}
\mathbb{E}(Z|(X+Z)=x) &= \mu_2 + (x-\mu_1-\mu_2)\frac{(\sigma_N^{\mathbb{Q}})^2 + \rho\sigma_N^{\mathbb{Q}}\sigma_I^{\mathbb{Q}}}{V} \\
\text{var}(Z|(X+Z)=x) &= (\sigma_N^{\mathbb{Q}})^2 - \frac{((\sigma_N^{\mathbb{Q}})^2 + \rho\sigma_N^{\mathbb{Q}}\sigma_I^{\mathbb{Q}})^2}{V} \\
&= \frac{(1-\rho^2)(\sigma_N^{\mathbb{Q}})^2(\sigma_I^{\mathbb{Q}})^2}{V}
\end{aligned}$$

Matching the conditional moments under both measures we get the system (3.43)-(3.45).

We next prove that for all values of $\rho \in (-1,1)$ there exist μ_1, μ_2, $\sigma_I^{\mathbb{Q}}$ and $\sigma_N^{\mathbb{Q}}$ such that the system (3.43)-(3.45) has a solution. Dividing (3.44) by (3.43) we get

$$(1-\rho^2)\sigma_N^{\mathbb{Q}}(\sigma_I^{\mathbb{Q}})^2 - B\rho\sigma_I^{\mathbb{Q}} - B\sigma_N^{\mathbb{Q}} = 0,$$

which is a quadratic equation for $\sigma_I^{\mathbb{Q}}$. In order for $\sigma_I^{\mathbb{Q}}$ to be positive we have to take the positive root of the equation and we obtain

$$\sigma_I^{\mathbb{Q}} = \frac{B\rho + \sqrt{B^2\rho^2 + 4(1-\rho^2)B(\sigma_N^{\mathbb{Q}})^2}}{2(1-\rho^2)\sigma_N^{\mathbb{Q}}}$$

Inserting $\sigma_I^{\mathbb{Q}}$ into (3.43) we get that the following equation has to hold ($x = \sigma_N^{\mathbb{Q}}$)

$$\begin{aligned}
L(x) &= R(x) \qquad (3.46) \\
L(x) &= x^2 + \rho\frac{B\rho + \sqrt{B^2\rho^2 + 4(1-\rho^2)Bx^2}}{2(1-\rho^2)} \\
R(x) &= A(x^2 + (\sigma_I^{\mathbb{Q}})^2(x) + 2\rho x\sigma_I^{\mathbb{Q}}).
\end{aligned}$$

The equation (3.46) can not be solved explicitly, so we only show that the solution exist. We know the following behavior of L and R:

$$\begin{aligned}
\lim_{x\searrow 0} L(x) &< \infty \\
\lim_{x\nearrow\infty} \frac{L(x)}{x^2} &= 1 \\
\lim_{x\searrow 0} R(x) &= \infty \\
\lim_{x\nearrow\infty} \frac{R(x)}{x^2} &= A < 1
\end{aligned}$$

It follows that the system (3.46) always has a solution.

The mean of $X + Z$ under the measure $\mathbb{Q}$ is according to (3.45) given by

$$\mu_1 + \mu_2 \;=\; -\mu + \mu_2 \frac{\sigma_I^2 + \sigma_N^2}{\sigma_N^2}.$$

□

The following Proposition proves that the audit costs are linear (affine) provided that the costs of the auditor are linear in its effort.

Proposition 3.A.9. *Let the assumptions of Section 3.2 hold with the exception of the structural form of c and let the unknown variable $\sigma_N \in \{\sigma_N^H, \sigma_N^L\}$ be known to the auditor but unknown to the firm, see the discussion in Section 3.2.3. Then the audit costs C_H, C_L connected to different levels of σ_N are given by*

$$C_H \;=\; D\sigma_N^H R_H \tag{3.47}$$

$$C_L \;=\; D\sigma_N^L R_L + D(\sigma_N^H - \sigma_N^L) R_H. \tag{3.48}$$

Proof. We formulate the firm's contracting problem as

$$\max_{C_H, C_L, R_H, R_L} \mathbb{E}\left[\mu - C_i - \alpha(\sigma_I^2 + \sigma_N^{i2} 2^{-2R_i})\right] \tag{3.49}$$

subject to the auditor's incentive compatibility constratints

$$C_H - D\sigma_N^H R_H \;\geq\; C_L - D\sigma_N^H R_L \tag{3.50}$$

$$C_L - D\sigma_N^L R_L \;\geq\; C_H - D\sigma_N^L R_H \tag{3.51}$$

and participation constraints

$$C_H - D\sigma_N^H R_H \;\geq\; 0 \tag{3.52}$$

$$C_L - D\sigma_N^L R_L \;\geq\; 0 \tag{3.53}$$

It is easy to show that

$$C_L - D\sigma_N^L R_L \geq C_H - D\sigma_N^L R_H \geq C_H - D\sigma_N^H R_H \geq 0 \tag{3.54}$$

where the middle inequality is inferred from the incentive compatibility constraint (3.51), the second one since $\sigma_N^H \geq \sigma_N^L$ and the third one from the participation constraint (3.52). Therefore (3.52) holds with equality and we arrive at (3.47). Using (3.47) in (3.51) we get that

$$C_L - D\sigma_N^L R_L \geq C_H - D\sigma_N^L R_H = D(\sigma_N^H - \sigma_N^L) R_H > 0$$

and therefore we can not have $C_L - D\sigma_N^L R_L = 0$ but rather (3.48), an affine function of R_L. It therefore follows that the cost function is either linear or affine, not very differnt than the assumed $c(R)$. □

Proof. (of Proposition 3.2.3.) The proof and the problem setting follows Vayanos (1999) closely. Let $\mathbb{F}_t = \sigma\{S(1), \ldots, S(t)\}$ be σ-algebra generated by the white noise process S. We write $\mathbb{E}[\cdot|\mathbb{F}_t] = \mathbb{E}_t[\cdot]$. We conjecture that the demand $x(t)$ for the risky asset at time t is given by

$$x(t) = AX(t) - Bp(t) + Ce(t) + D \tag{3.55}$$

with A, B, C and D constants and assume the following functional form for p

$$p(t) = aX(t) + ce(t) + d, \tag{3.56}$$

with a, c and d constants. From (3.56) it follows that

$$\begin{aligned} p(t+1) - p(t) &= a(X(t+1) - X(t)) + c(e(t+1) - e(t)) \\ &= a((\rho - 1)X(t) + S(t+1)) + cx(t). \end{aligned}$$

Let V be the representative's agent value function, i.e.

$$V(W, X, e, t) = \max_{\{c(s),x(s)\}_{s\geq t}} \mathbb{E}_t \left[-\sum_{s=t}^{\infty} \beta^s e^{-\alpha c(s)} \right]$$

subject to constraints (3.8)-(3.9). In line with Vayanos (1999) we conjecture that the value function is given by

$$V(W, X, e, t) = -\exp(-\alpha(HX(t)e(t) + Fe(t)^2 + GW(t) + L)) \tag{3.57}$$

for some constants H, F, G and L. We proceed to compute $\mathbb{E}_t[V(W, X, e, t+1)]$. We divide the calculation into two parts:

$$\begin{aligned} \mathbb{E}_t[e^{-\alpha GW(t+1)}] &= e^{-\alpha G(1+\delta)(W(t)-c(t))} \mathbb{E}_t \left[e^{-\alpha Ge(t)X(t+1) - \alpha Gp(t+1)x(t)} \right] \\ &= e^{-\alpha G((1+\delta)[W(t)-c(t)]-e(t)\rho X(t))} \mathbb{E}_t \left[e^{-\alpha G(x(t)(p(t)+a(\rho-1)X(t)+aS(t+1)+cx(t))+e(t)S(t+1))} \right] \\ &= e^{-\alpha G((1+\delta)[W(t)-c(t)]-e(t)\rho X(t)+x(t)(p(t)+a(\rho-1)X(t)+cx(t)))} \\ &\qquad \mathbb{E}_t \left[e^{-\alpha G\{ax(t)+e(t)\}S(t+1)} \right] \end{aligned}$$

and

$$\mathbb{E}_t \left[e^{-\alpha HX(t+1)e(t+1)} \right] = e^{-\alpha H\rho X(t)e(t+1)} \mathbb{E}_t[e^{-\alpha HS(t+1)e(t+1)}].$$

Putting all together we get

$$\begin{aligned} \overline{V}(W, X, e, t) &= \mathbb{E}_t \left[V(W, X, e, t+1) \right] \\ &= -\exp\{-\alpha G[(1+\delta)[W(t) - c(t)] - e(t)\rho X(t) + \\ &\qquad x(t)(p(t) + a(\rho - 1)X(t) + cx(t))] - \alpha H\rho X(t)e(t+1) \\ &\qquad + \frac{1}{2} \left(-\alpha G(ax(t) + e(t)) - \alpha He(t+1)\right)^2 \sigma^2 \\ &\qquad - \alpha\mu(G(ax(t) + e(t)) + H(x(t) + e(t)))\} . \end{aligned} \tag{3.58}$$

The Bellman equation for V can therefore be written as

$$\begin{aligned} V(W, X, e, t) &= \max_{c(t),x(t)} \left\{ -e^{-\alpha c(t)} + \beta\mathbb{E}_t[V(W, X, e, t+1)] \right\} \\ &= \max_{c(t),x(t)} \left\{ -e^{-\alpha c(t)} + \beta\overline{V}(W, X, e, t) \right\} \end{aligned} \tag{3.59}$$

Differentiating (3.58) with respect to $x(t)$ gives us

$$\begin{aligned} 0 &= -\alpha G[p(t) + a(1-\rho)X(t) + cx(t)] - \alpha Gcx(t) - \alpha H\rho X(t) \\ &\quad +\sigma^2\alpha^2(Ga+H)(G(ax(t)+e(t)) + H(e(t)+x(t))) \\ &\quad -\alpha\mu(Ga+H) \end{aligned}$$

which can be rewritten as

$$L_0 p(t) + L_1 x(t) + L_2 e(t) + L_3 X(t) + L_4 = 0, \tag{3.60}$$

where the coefficients L_i, $i = 0, \ldots, 4$ are as follows:

$$\begin{aligned} L_0 &= -\alpha G \\ L_1 &= -2\alpha Gc + \alpha^2\sigma^2(Ga+H)^2 \\ L_2 &= \alpha^2\sigma^2(Ga+H)(G+H) \\ L_3 &= -\alpha Ga(1-\rho) - \alpha H\rho \\ L_4 &= -\alpha\mu(Ga+H) \end{aligned}$$

This confirms the structural form of prices (3.56).

Differentiating the expression in the maximum of equation (3.59) with respect to $c(t)$ gives us

$$-\alpha e^{-\alpha c(t)} + \beta\alpha G(1+\delta)\overline{V}(t) = 0$$

or differently

$$\begin{aligned} &\alpha c(t) - \alpha G\left\{(1+\delta)[W(t)-c(t)] - e(t)\rho X(t) + x(t)\left[ce(t+1) + d + a\rho X(t)\right]\right\} \\ &\quad -\alpha H\rho X(t)e(t+1) + \frac{1}{2}\left\{-\alpha G\left[ax(t)+e(t)\right] - \alpha He(t+1)\right\}^2\sigma^2 \\ &\quad +\log(\beta G(1+\delta)) = 0 \end{aligned} \tag{3.61}$$

Putting together (3.57), (3.58), the Bellman equation (3.59) and (3.61) we get that

$$G = \frac{\delta}{1+\delta}.$$

Using the envelope theorem with respect to $X(t)$ in equation (3.59) we compute the fraction $\frac{G}{H}$:

$$\frac{\partial V}{\partial X} = V(W,X,e,t)(-\alpha He(t))$$

and

$$\frac{\partial \overline{V}}{\partial X} = \overline{V}(\alpha G\rho e(t) - \alpha Ga(\rho-1)x(t) - \alpha H\rho(x(t)+e(t))).$$

Equating the terms at $e(t)$ in the previous two equations we get

$$-\alpha H = \beta(\alpha G\rho - \alpha H\rho)$$

from where it follows that

$$\frac{G}{H} = 1 - \frac{1}{\beta\rho} < 0. \tag{3.62}$$

Since there is only a representative investor in the market, the equilibrium conditions require that $e(t) = 1$ and $x(t) = 0$ for all t. Using the equilibrium conditions in (3.60) we obtain

$$p(t) = -\frac{L_2 + L_4}{L_0} - \frac{L_3}{L_0}X(t). \tag{3.63}$$

Matching expressions (3.56) and (3.63) we get the following relationship for a:

$$\begin{aligned} a &= -\frac{L_3}{L_0} \\ &= \frac{\alpha Ga(1-\rho) + \alpha H\rho}{-\alpha G} \\ &= -a(1-\rho) - \frac{H}{G}\rho \end{aligned}$$

from where it follows that

$$a = -\frac{\rho}{2-\rho}\frac{H}{G}$$

and taking into account the relationship (3.62) we get that

$$a = \frac{\rho}{2-\rho}\frac{\beta\rho}{1-\beta\rho} > 0.$$

The firms maximize the future discounted value of share prices, i.e. they choose per period auditing investment r as to solve (3.11). For simplicity[7] we assume that $\mathbb{E}(X(0)) = \mu_0 = \mu - Cr^*$. From the recursive relation (3.10) for $X(t)$ we get that $\mathbb{E}(X(t)) = (\mu - Cr^*)\frac{1-\rho^{t+1}}{1-\rho}$. Therefore

$$\begin{aligned} \sum_{s=1}^{\infty} \mathbb{E}(\gamma^s p(s)) &= \sum_{s=1}^{\infty} \gamma^s \left[a\mathbb{E}(X(s)) + c'(\sigma^2(r^*))\right] \\ &= \frac{a(\mu - Cr^*)}{1-\rho} \sum_{s=1}^{\infty} [\gamma^s(1-\rho^s)] + c'(\sigma^2(r^*)) \sum_{s=1}^{\infty} \gamma^s \\ &= \frac{a(\mu - Cr^*)}{1-\rho} \left[\frac{\gamma}{1-\gamma} - \frac{\rho\gamma}{1-\gamma}\right] + \frac{\gamma}{1-\gamma} c'(\sigma^2(r^*)) \\ &= \frac{\gamma}{1-\gamma} \left[a(\mu - Cr^*) + c'(\sigma^2(r^*))\right], \end{aligned}$$

where

$$\begin{aligned} c'(\sigma^2(r^*)) &= -\frac{L_2 + L_4}{L_0} \\ &= -\frac{\alpha^2\sigma^2(Ga + H) - \alpha\mu(Ga + H)}{-\alpha G} \\ &= \alpha\sigma^2\left(a + \frac{H}{G}\right)(G + H) - \mu\left(a + \frac{H}{G}\right). \end{aligned}$$

[7] The general case is not much more difficult than this and it does not offer any significantly different economic insights.

We have

$$\begin{aligned}
a + \frac{H}{G} &= \frac{\beta\rho}{1-\beta\rho}\frac{2(\rho-1)}{2-\rho} < 0 \\
H + G &= G\frac{\beta\rho}{\beta\rho - 1} + G \\
&= G\frac{2\beta\rho - 1}{\beta\rho - 1} \\
&= \frac{\delta}{1+\delta}\frac{2\beta\rho - 1}{\beta\rho - 1}.
\end{aligned}$$

Putting everything together we can write

$$\begin{aligned}
\sum_{s=1}^{\infty}\mathbb{E}(\gamma^s p(s)) &= \frac{\gamma}{1-\gamma}\left[a(\mu - Cr^*) + \left(a + \frac{H}{G}\right)\alpha\sigma^2(r^*)(G+H) - \mu\left(a + \frac{H}{G}\right)\right] \\
&= \frac{\gamma}{1-\gamma}\left(-\frac{H}{G}\right)\left[\mu + a\frac{G}{H}Cr^* + \left(-\frac{G}{H}\right)\alpha\sigma^2(r^*)\left(a + \frac{H}{G}\right)(G+H)\right] \\
&= \frac{\gamma}{1-\gamma}\left(-\frac{H}{G}\right)\left[\mu - C_n r^* + \alpha_n\sigma^2(r^*)\right]
\end{aligned}$$

The optimization problem coincides with the one-period problem (3.3) where the new cost and risk aversion parameters

$$\begin{aligned}
C_n &= -a\frac{G}{H}C = -\frac{\rho}{2-\rho}\frac{H}{G}\left(-\frac{G}{H}\right)C = \frac{\rho}{2-\rho}C \\
\alpha_n &= \alpha\left(-1 - a\frac{G}{H}\right)(G+H) \\
&= \alpha\frac{\delta}{1+\delta}\frac{2\beta\rho-1}{\beta\rho-1}\left(-1 + \frac{\rho}{2-\rho}\right) \\
&= \alpha\frac{\delta}{1+\delta}\frac{2\beta\rho-1}{\beta\rho-1}\frac{2(\rho-1)}{2-\rho}
\end{aligned}$$

The optimal per-period choice of r by the firm satisfies the same structural form as in the one-period case with the cost and the risk aversion parameters changed as indicated above. □

Proposition 3.A.10. *Let the same conditions as in Proposition 3.2.3 hold with the exception that the firm's objective function is*

$$\max_{r\geq 0}\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\sum_{s=1}^{t}p(s)\right]. \tag{3.64}$$

If $\rho > 2/3$ then the results of Proposition 3.2.3 still apply in this setting and the new per-period auditing costs and risk aversion parameters are

$$\begin{aligned}
C_n &= \frac{\rho}{3\rho - 2}C \\
\alpha_n &= \frac{2(1-\rho)^2}{3\rho-2}\frac{\delta}{1+\delta}\frac{2\beta\rho-1}{1-\beta\rho}\alpha
\end{aligned}$$

Proof. The only thing that differs in the proof of Proposition 3.2.3 from this setting is the firm analysis. The firms maximize the long-run average of stock prices, i.e. they choose per period auditing investment r as to solve (3.64). We assume as before that $\mathbb{E}(X(0)) = \mu_0 = \mu - Cr^*$. The firm's optimizing function therefore reads

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\sum_{s=1}^{t}\mathbb{E}(p(s)) &= \lim_{t\to\infty}\frac{1}{t}\sum_{s=1}^{t}\left[a\mathbb{E}(X(s)) + c'(\sigma^2(r^*))\right] \\
&= \lim_{t\to\infty}\frac{1}{t}\frac{(\mu - Cr^*)}{1-\rho}\sum_{s=1}^{t}\left[a(1-\rho^{s+1})\right] + c'(\sigma^2(r^*)) \\
&= \frac{a(\mu - Cr^*)}{1-\rho}\lim_{t\to\infty}\frac{1}{t}\sum_{s=1}^{t}(1-\rho^s) + c'(\sigma^2(r^*)) \\
&= \frac{a(\mu - Cr^*)}{1-\rho}\lim_{t\to\infty}\left(1-\rho^2\frac{1-\rho^t}{(1-\rho)t}\right) + c'(\sigma^2(r^*)) \\
&= \frac{a(\mu - Cr^*)}{1-\rho} + c'(\sigma^2(r^*)) \\
&= \frac{a}{1-\rho}\left((\mu - Cr^*) + \frac{1-\rho}{a}c'(\sigma^2(r^*))\right) \\
&= \frac{a}{1-\rho}\left[\mu - Cr^* + \frac{1-\rho}{a}\left(\alpha\sigma^2(r)\left\{a+\frac{H}{G}\right\}(G+H) - \mu\left(a+\frac{H}{G}\right)\right)\right] \\
&= \frac{a}{1-\rho}\left[\mu\rho - \mu\frac{H}{G}\frac{1-\rho}{a} - Cr^* + \frac{1-\rho}{a}\alpha\left\{a+\frac{H}{G}\right\}(G+H)\sigma^2(r^*)\right] \\
&= \frac{a}{1-\rho}\left[\mu\frac{3\rho-2}{\rho} - Cr^* + \frac{2(1-\rho)^2}{\rho}\frac{\delta}{1+\delta}\frac{2\beta\rho-1}{1-\beta\rho}\alpha\sigma^2(r^*)\right],
\end{aligned}
$$

where we have used the following relationships

$$
\begin{aligned}
\frac{1-\rho}{a} &= \frac{1-\rho}{\frac{\rho}{2-\rho}\frac{\beta\rho}{1-\beta\rho}} \\
&= \frac{(1-\rho)(2-\rho)(1-\beta\rho)}{\beta\rho^2} \\
\frac{H}{G}\frac{1-\rho}{a} &= \frac{(1-\rho)(2-\rho)}{\rho} \\
\rho - \frac{H}{G}\frac{1-\rho}{a} &= \frac{3\rho-2}{\rho} \\
\frac{1-\rho}{a}\left(a+\frac{H}{G}\right)(G+H) &= \frac{(1-\rho)(2-\rho)(1-\beta\rho)}{\rho^2\beta}\frac{\beta\rho}{1-\beta\rho}\frac{2(\rho-1)}{2-\rho}\frac{\delta}{1+\delta}\frac{2\beta\rho-1}{\beta\rho-1} \\
&= \frac{2(1-\rho)^2}{\rho}\frac{\delta}{1+\delta}\frac{2\beta\rho-1}{1-\beta\rho}
\end{aligned}
$$

□

Proof. (of relationship (3.14).) The first inequality follows from

$$\frac{\rho}{2-\rho} - 1 = \frac{2(\rho-1)}{2-\rho} < 0.$$

To prove the second we write

$$\frac{\rho}{3\rho - 2} - 1 = \frac{2(1-\rho)}{3\rho - 2} > 0$$

by assumption that $\rho > 2/3$. □

Proof. (of Propositions 3.2.4 and 3.2.5.) We define the firm's auditing profit function

$$l(R) = \alpha\sigma_N^2(1 - 2^{-2R}) - CR, \tag{3.65}$$

i.e. auditing is beneficial to the firm if $l(R) > 0$ and destructive of firm's value otherwise. We also define the firm's zero-profit condition as the value R_1 which solves the equation $l(R_1) = 0$, see equation (3.5). The auditor's profit function is

$$m(R) = CR - D\sigma_N^2.$$

Since the auditor's profit function is increasing in R we have that $m(R^*) > m(0)$ where R^* is defined in (3.4). The auditor accepts the offer if and only if

$$CR^* > D\sigma_N^2$$

or equivalently if (3.15) holds.

To prove part b) in the Proposition let R_1 be the largest solution to $l(R_1) = 0$. It is easy to prove that $l(R) < 0$ for $R > R_1$ and $l(R) \geq 0$ for $0 \leq R \leq R_1$. If $R_1 = 0$ then no audit contract is ever accepted by the firm. Let us therefore assume that $R_1 > R^* > 0$, where R^* is defined in (3.4). It is also obvious that the auditing effort by both firms is the same since by deviating the firm can only increase its chances of winning the contract. If both firms bid $R > R^*$ and $\frac{1}{2}CR - D\sigma_N^2 > 0$ then both firms have the incentive to propose the auditing closer to R^*. The same happens when firms start with $R < R^*$. The proposition follows. □

Proof. (of Proposition 3.2.6.) We first solve the optimization problem of the auditor, i.e.

$$\max_{b'} \mathbb{E}[(Cr - D\sigma_N^2) \cdot \mathbb{1}(S_2 \leq b')|S_2] = \mathbb{1}(S_2 \leq b')(Cr - DS_2).$$

The maximum is obtained for $b' = \frac{Cr}{D}$.

We now proceed by solving the optimization problem of the firm, i.e.

$$\begin{aligned} &\max_r \mathbb{E}[(\mu - Cr - \alpha\sigma_I^2 - \alpha\sigma_N^2 2^{-2r}) \cdot \mathbb{1}(S_2 \leq b') + (\mu - \alpha\sigma_I^2 - \alpha\sigma_N^2) \cdot \mathbb{1}(S_2 > b')] \\ &\quad = \quad (\mu - Cr - \alpha\sigma_I^2 - \alpha\sigma_N^2 2^{-2r})\, \mathbb{P}(S_2 \leq b') + (\mu - \alpha\sigma_I^2 - \alpha\sigma_N^2)\, \mathbb{P}(S_2 > b') \\ &\quad = \quad (\mu - \alpha\sigma_I^2 - \alpha\sigma_N^2) + \mathbb{P}(S_2 \leq b')(-Cr + \alpha\sigma_N^2(1 - 2^{-2r})) \end{aligned}$$

□

Proof. (of Proposition 3.3.1.) The relationship between arithmetic and geometric mean together with relationships (3.18) and (3.19) confirm that $v(A) + v(L_I) \leq v(A, L_I)$, $v(A, L_I) + v(L_J) \leq v(A, L_{I\cup J})$, $v(L_I) + v(L_J) \leq v(L_{I\cup J})$, where I and J are disjunct subsets of $\{1, \ldots, n\}$. This proves the superadditivity of function v.

We next compute the Shapley value of the auditor, i.e.

$$\varphi_A = \sum_{S\subset\{1,\ldots,n\}} \frac{s!(n-s-1)!}{n!}(v(a,S)-v(S)).$$

Let

$$\begin{aligned} AD(S) &= v(a,S)-v(S) \\ &= \left(-C|S|r^*(|S|)+\alpha\sum_{i\in S}\sigma_{Ni}^2-\alpha|S|\prod_{i\in S}\frac{\sigma_{Ni}^{2/|S|}}{2^{2r^*}}\right), \end{aligned} \tag{3.66}$$

where $r^* = R^*/n$. Reinforcing the statement given at the beginning of this proof, we first show that (3.66) is positive for $r^* = 0$. By the inequality between arithmetic and geometric mean we have that ($s = |S|$)

$$\frac{1}{s}\sum_{i\in S}\sigma_i^2 \geq \left(\prod_{i\in S}\sigma_i^2\right)^{1/s},$$

where $s = |S|$. Therefore it is even more the case that

$$\frac{1}{s}\sum_{i\in S}\sigma_i^2 \geq \prod_{i\in S}\frac{\sigma_i^{2/s}}{2^{2r^*}}$$

from where it follows that

$$\sum_{i\in S}\sigma_i^2 - s\prod_{i\in S}\frac{\sigma_i^{2/s}}{2^{2r^*}} \geq 0.$$

In the following we use the facts that

$$\begin{aligned} \sum_{S\subset\{1,\ldots,n\}}\frac{s!(n-s-1)!}{n!} &= \sum_{s=1}^{n}\frac{s!(n-s-1)!}{n!}\binom{n}{s} \\ &= \sum_{s=1}^{n}\frac{1}{n-s} = \sum_{s=1}^{n}\frac{1}{s} \\ &= H(n) \\ \sum_{S\subset\{1,\ldots,n\}}\frac{s!(n-s-1)!}{n!}s &= \sum_{s=1}^{n}\frac{s}{n-s} \\ &= \sum_{s=1}^{n}\frac{n-s}{s} = nH(n)-n \\ &= n(H(n)-1), \end{aligned}$$

where $H(n)$ is the n-th Harmonic number. Therefore

$$\begin{aligned}
\sum_{S\in\{1,\dots,n\}} \frac{s!(n-s-1)!}{n!} Csr^* &= Cr^* n(H(n)-1) \\
\sum_{S\subset\{1,\dots,n\}} \frac{s!(n-s-1)!}{n!} \sum_{i\in S} \sigma_{Ni}^2 &= \left(\sum_{i=1}^{n} \sigma_{Ni}^2\right) \sum_{s=0}^{n-1} \binom{n-1}{s} \frac{s!(n-s-1)!}{n!} \\
&= \left(\sum_{i=1}^{n} \sigma_{Ni}^2\right) \frac{n-1}{n}.
\end{aligned}$$

The expression ($u = \frac{1}{2^{2r^*}}$)

$$\sum_{S\subset\{1,\dots,n\}} \frac{s!(n-s-1)!}{n!} s \prod_{i\in S} \frac{\sigma_{Ni}^{2/s}}{2^{2r^*}} = \sum_{S\subset\{1,\dots,n\}} \frac{s!(n-s-1)!}{n!} su^s \prod_{i\in S} \sigma_{Ni}^{2/s}$$

does not allow for analytical simplification. Putting all together gives us (3.20). Differentiating (3.19) for $S = \{1,\dots,n\}$ with respect to r gives us the condition for the optimal auditing effort r:

$$C = 2n \ \log 2 \ 2^{-2nr} \alpha \prod_{i\in\{1,\dots,n\}} \sigma_{Ni}^{2/n}. \tag{3.67}$$

Solving the equation (3.67) for r gives us (3.21). Inserting the r^* in (3.19) for $A = \{1,\dots,n\}$ gives us the result in (3.22). □

3.A.3 Merton's model

We assume a one period model where the total firm value $A(1)$ at the end of the period is distributed log-normally (similar to the CARA-normal framework in Section 3.2), i.e. $A_1 = A_0 \exp(\mu + \sigma N)$, where N is the standard normal random variable and A_0 is the firm's asset value at the beginning of the period. Under the risk-neutral pricing measure[8] the value of the company at the end of the period is given by $A_1 = A_0 \exp\left(-\frac{1}{2}\sigma^2 + \sigma N\right)$. The agent (debtholder or shareholder) chooses the level of auditing effort R. By doing this, the initial value of the company is reduced to $A_0 - c(R)$ and the economic noise volatility is reduced from σ_N to $\sigma_N 2^{-R}$, as discussed in Section 3.2. Company's liabilities are zero-coupon debt with principal P and equity. The value of debt and equity under auditing is determined by arbitrage arguments, i.e.

$$\begin{aligned}
D(R) &= \mathbb{E}(\min(P, A_1(R))) \\
&= P\,\mathbb{P}\left((A_0 - c(R)) \exp\left(-\frac{1}{2}\sigma^2(R) + \sigma(R)N\right) > P\right) \\
&\quad + \int_{-\infty}^{y^U(R)} (A_0 - c(R)) \exp\left(-\frac{1}{2}\sigma^2(R) + \sigma(R)y\right) n(y)\,\mathrm{d}y \\
&= PN(-y^U(R)) + (A_0 - c(R))N(y^U(R) - \sigma(R)),
\end{aligned} \tag{3.68}$$

[8] We assume as in the previous sections that the riskless interest rate is normalized to 0.

where n, N are the probability density and distribution function of the standard normal random variable and $y^U(R) = \frac{1}{\sigma(R)}\left(\log\left(\frac{P}{A_0 - c(R)}\right) + \frac{1}{2}\sigma^2(R)\right)$. The value of the equity is then given by $S = A_0 - c(R) - D$. Debtholder (shareholder) choose the auditing effort R_D (R_S) in order to maximize the value of debt (equity), i.e. $\{R_D, R_S\} = \text{argmax}_{R\geq 0}\{D(R), S(R)\}$. Debt yield (stock prices) under auditing are denoted by Y_A (S_A) and under the lowest audit regime as Y_{NA} (S_{NA}). It is trivial to prove that $Y_A \leq Y_{NA}$, the reason being that Y_A is the yield that maximizes the debt value (3.68).

3.A.4 Auditing game with incomplete information for both the auditor and the firm

We next address the issue of external noise variance parameter σ_N. In Section 3.2 we assumed that the parameter was known by both the firm and the auditor perfectly. Here we relax this restrictive assumption. When employing an external auditor the shareholders do not know if the increase in firm value comes from auditing procedure by reducing σ_N^2 or through the reduction of the intrinsic cash flow volatility σ_I^2, see (3.3). We model this behavior in a framework of signalling games, see Fudenberg and Tirole (1991), Chapter 3. There exist two players, the shareholders of the firm (called only the firm), player 1 and the auditor, player 2. All variables in the remaining part of this section with index 1 and 2 refer respectively to the firm and the auditor. The game proceeds in the following fashion. First the firm receives a signal S_1 about σ_N^2. The signal is unbiased. The firm then proposes a contract of auditing effort level $r = s_1(S_1)$ to the auditor. The auditor receives independently of the shareholders a signal S_2 regarding σ_N^2 and decides whether to accept or decline the audit offer. If the auditor declines, it earns zero profit. If the auditor accepts the auditing contract, it collects the audit fee and incurs a cost $D\sigma_N^2$. The auditor's response function $s_2(S_2)$ therefore takes only values in $\{\text{Accept}, \text{Decline}\}$.

The signal S_2 of the auditor can come from a multitude of sources. The auditors usually audit a large portfolio of firms and have knowledge of the amount of noise in the industry sector that the firm operates. Furthermore they can compare firm share prices to those of other similar ones and obtain more or less reliable information about the noise level. The firm on the other hand does not know σ_I (and therefore σ_N) perfectly too. They can obtain a time series of its share prices and from it by taking a long-run average infer the average value of σ_N. The remaining part of firm excess volatility over its long-run average is attributed to economic noise. We refer to this as the firm's signal S_1.

The game matrix for this signalling game is given in Table 3.3. The following proposition char-

		Auditor	
		Accept	Decline
Firm	r	$(\mu - Cr - \alpha\sigma_I^2 - \alpha\sigma_N^2 2^{-2r}, Cr - D\sigma_N^2))$	$(\mu - \alpha\sigma_I^2 - \alpha\sigma_N^2, 0)$

Table 3.3: The game matrix for the signalling game between the firm and the auditor. The first entry in the parenthesis is the payoff to the firm and the second to the auditor for both actions (Accept or Decline) of the auditor. The allowed values for $r \in \mathbb{R}_+$.

acterizes the actions of both players in this setting.

Proposition 3.A.11. *Let* $\mathbb{E}(\sigma_N^2|S_1, S_2) = aS_1 + bS_2$ *where* $a, b \geq 0$ *and* $a + b = 1$. *In the setting* *above the firm proposes a contract of auditing level $s_1^*(S_1)$ which minimizes*

$$s_1^* \in \operatorname*{argmin}_{s_1} CN(-b'(s_1))s_1 + \alpha a N(-b'(s_1))S_1 2^{-2s_1}\left(1 + \frac{N(b'(s_1))}{N(-b'(s_1))}2^{2s_1}\right) \tag{3.69}$$

where $b'(s_1) = \frac{Cs_1 - DaS_1(s_1)}{Db}$ and $N(x) = \mathbb{P}(S_2 \le x)$. The auditor's decision to accept the auditing offer has a cutoff value $b'(s_1)$ with signals S_2 below $b'(s_1)$ eliciting an acceptance of the auditing procedure, i.e.

$$s_2^*(S_2) = \begin{cases} \textit{Accept} & S_2 \le b'(s_1) \\ \textit{Decline} & S_2 > b'(s_1) \end{cases}$$

Proof. Let the strategy of the firm be a function $S_1 \mapsto s_1(S_1)$, where S_1 is the signal that the firm receives about the noise component σ_N^2. The strategy of the auditor is a cutoff decision, i.e. accept the auditing procedure if $S_2 < b(s_1)$, and decline it otherwise. Given this strategy profile we first analyze the decision of the auditor. The auditor maximizes

$$\mathbb{1}(S_2 < b(s_1)) \cdot \left(Cs_1 - D\mathbb{E}(\sigma_N^2|s_1, S_2)\right). \tag{3.70}$$

Using the assumption $\mathbb{E}(\sigma_N^2|s_1, S_2) = \mathbb{E}(\sigma_N^2|S_1, S_2) = aS_1 + bS_2$ the auditor's profit in the case when the auditing contract is accepted is

$$Cs_1 - DaS_1 - DbS_2,$$

and 0 otherwise. Hence, the auditing contract is accepted if and only if

$$S_2 < b'(s_1).$$

We now analyze the contract setting by the firm which sets s_1 so that it is a best response to the auditor, i.e. it maximizes (minimizes)

$$\begin{aligned} s_1(S_1) &\in \operatorname*{argmax}_{s_1} N(b'(s_1))\mathbb{E}(\mu - \alpha\sigma_I^2 - \alpha\sigma_N^2|S_1) \\ &\qquad +N(-b'(s_1))\mathbb{E}(\mu - Cs_1 - \alpha\sigma_I^2 - \alpha\sigma_N^2 2^{-2s_1}|S_1), \\ &\in \operatorname*{argmax}_{s_1} -N(b'(s_1))\mathbb{E}(\alpha\sigma_N^2|S_1) - Cs_1N(-b'(s_1)) - N(-b'(s_1))\mathbb{E}(\alpha\sigma_N^2 2^{-2s_1}|S_1) \\ &\in \operatorname*{argmin}_{s_1} N(b'(s_1))\alpha a S_1 + Cs_1N(-b'(s_1)) + N(-b'(s_1))\alpha 2^{-2s_1} aS_1 \end{aligned}$$

with respect to s_1 where we used the equality

$$\begin{aligned} \mathbb{E}(\sigma_N^2|S_1) &= \mathbb{E}(\mathbb{E}(\sigma_N^2|S_1, S_2)|S_1) \\ &= \mathbb{E}(aS_1 + bS_2|S_1) \\ &= aS_1, \end{aligned}$$

since the signals S_1 and S_2 are independent. Rearranging the terms we obtain (3.69). $\square$

The condition that $\mathbb{E}(\sigma_N^2|S_1, S_2)$ is a linear function of S_1 and S_2 is a plausible condition in a normal setting, i.e. if (σ_N^2, S_1, S_2) are all jointly normal. The decision of the auditor is a cutoff decision, i.e. it accepts audit offers when the signal S_2 about the noise level volatility σ_N^2 compared to σ_N^2 is low enough implying low enough costs of auditing. The conditions of this sort are very common in the literature on global games and other games of imperfect information. The cutoff value of R in the complete information case is $\frac{D\sigma_N^2}{C}$ with values of $R < \frac{D\sigma_N^2}{C}$ eliciting a rejection of the auditing procedure. In the incomplete information case this cutoff value is

$$\frac{DbS_2 + DaS_1(s_1)}{C}. \tag{3.71}$$

Two similarities between the two cases are obvious. First the known value of σ_N^2 in the complete information case is replaced by the signal S_2 about it. Secondly, the auditing costs D are replaced by smaller costs Db which would imply that auditing is rejected less often. This is not the case due to the second factor in the numerator of (3.71) which increases the cutoff value: Since the firm's optimal auditing contract effort level s_1 is an increasing function of S_1, so is the inverse function $S_1(s_1)$.

The decision of the firm is a more complicated function of its signal S_1. The contract offered by the firm in the incomplete information setting resembles the contract in the perfect information case in equation (3.3) which we rewrite here in terms of a minimizing problem for coherence:

$$\begin{aligned} R &\in \operatorname*{argmax}_R -CR - \alpha\sigma_N^2 2^{-2R} \\ &\in \operatorname*{argmin}_R CR + \alpha\sigma_N^2 2^{-2R}. \end{aligned} \tag{3.72}$$

Comparing optimal firm decisions (3.69) and (3.72) in the incomplete and complete information setting we see that the signal S_1 takes the place of the actual noise variance σ_N^2 in the full information setting. The auditing costs of the incomplete information game are virtually lowered - $CN(-b'(s_1))$ instead of full auditing costs C - in comparison to the full information economy. If this was the only change the demand for auditing in the this framework would be actually higher: lowered auditing costs generally result in higher demand for auditing R. Unfortunately this is not the only change the firm is concerned with. The risk-aversion parameter in the incomplete information setting (3.69) is the complete information α multiplied by

$$aN(-b'(s_1))\left(1 + \frac{N(b'(s_1))}{N(-b'(s_1))}2^{2s_1}\right) = a(N(-b'(s_1)) + N(b'(s_1))2^{2s_1}).$$

Two forces are at work here. The parameter $a \le 1$. On the other hand the second factor in equation above is in the normal setting (which is appropriate for the assumptions made above)

$$N(-b'(s_1)) + N(b'(s_1))2^{2s_1} \ge 1,$$

since $N(-b'(s_1)) + N(b'(s_1)) = 1$ and $2^{2s_1} \ge 1$ (the auditing contract $s_1 \ge 0$). For a sufficiently close to 1 we have that $aN(-b'(s_1))\left(1 + \frac{N(b'(s_1))}{N(-b'(s_1))}\right) > 1$. The incomplete information setting resembles that of the incomplete information but with reduced marginal costs of auditing C and increased level of risk aversion parameter α. We show in Proposition 3.A.12 in the Appendix the the incomplete information case converges to the complete information in Section 3.2 as the parameter uncertainty about the economic noise σ_N^2 decreases to zero, i.e. the firm's objective functions (3.69) and (3.72) coincide.

Proposition 3.A.12. *If the parameter uncertainty about σ_N^2 decreases to 0, that is $\mathbb{E}(\sigma_N^2|S_1, S_2) = S_1$ and $S_1 \to \sigma_N^2$ almost surely the firm's decisions in the incomplete information case (3.69) and the complete information case (3.72) coincide.*

Proof. As $a \to 1$ we have that $b \to 0$. Therefore $N(-b'(s_1))) = \mathbb{P}(S_2 < b'(s_1)) = \mathbb{P}(S_2 < \infty) = 1$ since $\lim_{b\to 0} b'(s_1) = \infty$. The result follows. $\square$

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